1111: Linear Algebra I

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Lecture 9

Previously on...

The determinant of an $n \times n$-matrix A is

$$\det(A) = \sum_{\sigma} \operatorname{sign}(\sigma) A_{i_1 j_1} A_{i_2 j_2} \cdots A_{i_n j_n} \; .$$

Here σ runs over all permutations of n elements, and $\begin{pmatrix} i_1 & i_2 & \dots & i_n \\ j_1 & j_2 & \dots & j_n \end{pmatrix}$ is some two-row representation of σ. The *sign*, or*signature* $\operatorname{sign}(\sigma)$ of a permutation σ is defined to be 1 if σ is even, and -1 if σ is odd.

For example, $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$. The formula for 3×3-matrices has six terms corresponding to the six permutations $1, 2, 3$ (even), $2, 1, 3$ (odd), $1, 3, 2$ (odd), $3, 1, 2$ (even), $2, 3, 1$ (even), and $3, 2, 1$ (odd). Usually we shall not use the formula directly, but rather rely on properties of determinants for computations.

Properties of determinants

- If three matrices A, A', and A'' have all rows except for the i-th row i in common, and the i-th row of A is equal to the sum of the i-th rows of A' and A'', then $\det(A) = \det(A') + \det(A'')$;
- if two matrices A and A' have all rows except for the i-th row in common, and the i-th row of A' is obtained from the i-th row of A by multiplying it by a scalar c, then $\det(A') = c\det(A)$;
- if two matrices A and A' have all rows except for the i-th and the j-th row in common, and A' is obtained from the A by swapping the i-th row with the j-th row, then $\det(A') = -\det(A)$;
- if two matrices A and A' have all rows except for the i-th row in common, and the i-th row of A' is obtained from the i-th row of A by adding a multiple of another row, then $\det(A') = \det(A)$;
- for each n, we have $\det(I_n) = 1$.

Properties of determinants

Effectively, these properties say that

- determinants are *multilinear functions* of their rows,
- determinants behave predictably with respect to elementary row operations.

Let us give an example of how this can be used:

$$\det\begin{pmatrix}1 & 2 & 2\\ 1 & 1 & 4\\ 1 & 2 & 8\end{pmatrix} \overset{(2)-(1),(3)-(1)}{=} \det\begin{pmatrix}1 & 2 & 2\\ 0 & -1 & 2\\ 0 & 0 & 6\end{pmatrix} =$$

$$(-1)\cdot 6\cdot\det\begin{pmatrix}1 & 2 & 2\\ 0 & 1 & -2\\ 0 & 0 & 1\end{pmatrix} \overset{(2)+2(3),(1)-2(3)}{=} -6\det\begin{pmatrix}1 & 2 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \overset{(1)-2(2)}{=}$$

$$-6\det\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = -6.$$

Properties of determinants

Let us prove the stated properties. In fact, they are quite easy to prove. Multilinearity is obvious: each of the terms in

$$\det(A) = \sum_{\sigma} \operatorname{sign}(\sigma) A_{i_1 j_1} A_{i_2 j_2} \cdots A_{i_n j_n}$$

contains exactly one term from each row, so if for two matrices all rows but one are the same, in the sum of their determinants we can collect the similar terms, and get the determinant where two rows are added. A similar but easier argument works for scalar multiples.

Change of sign under swapping rows is also clear: swapping rows corresponds to swapping two elements in the top row of the two-row notation of each permutation, so changes the sign.