1111: Linear Algebra I

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Lecture 7

Properties of the matrix product

Let us show that the matrix product we defined satisfies the following properties (whenever all matrix operations below make sense):

$$A \cdot (B + C) = A \cdot B + A \cdot C,$$
$$(A + B) \cdot C = A \cdot C + B \cdot C,$$
$$(c \cdot A) \cdot B = c \cdot (A \cdot B) = A \cdot (c \cdot B),$$
$$(A \cdot B) \cdot C = A \cdot (B \cdot C)$$

All these proofs can proceed in the same way: pick a "test vector" $\mathbf{x}$, multiply both the right and the left by it, and test that they agree. (Since we can take $\mathbf{x} = \mathbf{e}_j$ to single out individual columns, this is sufficient to prove equality).

For example, the first equality follows from

$$(A \cdot (B + C)) \cdot \mathbf{x} = A \cdot ((B + C) \cdot \mathbf{x}) = A \cdot (B \cdot \mathbf{x} + C \cdot \mathbf{x}) =$$
$$A \cdot (B \cdot \mathbf{x}) + A \cdot (C \cdot \mathbf{x}) = (A \cdot B) \cdot \mathbf{x} + (A \cdot C) \cdot \mathbf{x} = (A \cdot B + A \cdot C) \cdot \mathbf{x}$$

The identity matrix

Let us also define, for each n, the *identity* matrix I_n, which is an $n \times n$-matrix whose diagonal elements are equal to 1, and all other elements are equal to zero.

For each $m \times n$-matrix A, we have $I_m \cdot A = A \cdot I_n = A$. This is true because for each vector $\mathbf{x}$ of height p, we have $I_p \cdot \mathbf{x} = \mathbf{x}$. (The matrix I_p does not change vectors; that is why it is called the identity matrix). Therefore,

$$(I_m \cdot A) \cdot \mathbf{x} = I_m \cdot (A \cdot \mathbf{x}) = A \cdot \mathbf{x},$$
$$(A \cdot I_n) \cdot \mathbf{x} = A \cdot (I_n \cdot \mathbf{x}) = A \cdot \mathbf{x} \ .$$

Elementary matrices

Let us define elementary matrices. By definition, an elementary matrix is an $n \times n$-matrix obtained from the identity matrix I_n by one elementary row operation.

Recall that there were elementary operations of three types: swapping rows, re-scaling rows, and combining rows. This leads to elementary matrices S_{ij}, obtained from I_n by swapping rows i and j, $R_i(c)$, obtained from I_n by multiplying the row i by c, and $E_{ij}(c)$, obtained from the identity matrix by adding to the row i the row j multiplied by c.

Exercise. Write these matrices explicitly.

Main property of elementary matrices

Our definition of elementary matrices may appear artificial, but we shall now see that it agrees wonderfully with the definition of the matrix product.

Theorem. Let E be an elementary matrix obtained from I_n by a certain elementary row operation $\mathcal{E}$, and let A be some $n \times k$-matrix. Then the result of the row operation $\mathcal{E}$ applied to A is equal to $E \cdot A$.

Proof. By inspection, or by noticing that elementary row operations combine rows, and the matrix product $I_n \cdot A = A$ computes dot products of rows with columns, so an operation on rows of the first factor results in the same operation on rows of the product.

Invertible matrices

An $m \times n$-matrix A is said to be invertible, if there exists an $n \times m$-matrix B such that $A \cdot B = I_m$ and $B \cdot A = I_n$.

Why are invertible matrices useful? If a matrix is invertible, it is very easy to solve $A \cdot \mathbf{x} = \mathbf{b}$! Indeed,

$$B \cdot \mathbf{b} = B \cdot A \cdot \mathbf{x} = I_n \cdot \mathbf{x} = \mathbf{x} \ .$$

Some important properties:

- The equalities $A \cdot B = I_m$ and $B \cdot A = I_n$ can hold for at most one matrix B; indeed, if it holds for two matrices B_1 and B_2, we have

$$B_1 = B_1 \cdot I_m = B_1 \cdot (A \cdot B_2) = (B_1 \cdot A) \cdot B_2 = I_n \cdot B_2 = B_2 \ .$$

  Thus the matrix B can be called *the inverse of* A and be denoted A^{-1}.
- If both matrices A_1 and A_2 are invertible, and their product is defined, then A_1A_2 is invertible, and $(A_1A_2)^{-1} = A_2^{-1}A_1^{-1}$; indeed, for example

$$(A_1A_2)A_2^{-1}A_1^{-1} = A_1(A_2A_2^{-1})A_1^{-1} = A_1I_{m_2}A_1^{-1} = A_1A_1^{-1} = I_{m_1} \ .$$

  (As they say, "you put your socks on before putting on your shoes, but take them off after taking off your shoes").

Invertible matrices

Theorem. 1. An elementary matrix is invertible.
2. If an $m \times n$-matrix A is invertible, then $m = n$.
3. An $n \times n$-matrix A is invertible if and only if it can be represented as a product of elementary matrices.

Proof. 1. If $A = E$ is an elementary matrix, then for B we can take the matrix corresponding to the inverse row operation. Then $AB = I_n = BA$ since we know that multiplying by an elementary matrix performs the actual row operation.
2. Suppose that $m \neq n$, and there exist matrices A and B such that $A \cdot B = I_m$ and $B \cdot A = I_n$. Without loss of generality, $m > n$ (otherwise swap A with B). Let us show that $AB = I_m$ leads to a contradiction. We have $E_1 \cdot E_2 \cdots E_p \cdot A = R$, where R is the reduced row echelon form of A, and E_i are appropriate elementary matrices. Therefore,

$$R \cdot B = E_1 \cdot E_2 \cdots E_p \cdot A \cdot B = E_1 \cdot E_2 \cdots E_p \ .$$

Invertible matrices

From $R \cdot B = E_1 \cdot E_2 \cdots E_p$, we immediately deduce

$$R \cdot B \cdot (E_p)^{-1} \cdots (E_2)^{-1} \cdot (E_1)^{-1} = I_m \ .$$

But if we assume $m > n$, the last row of R is inevitably zero (there is no room for m pivots), so the last row of $R \cdot B \cdot (E_p)^{-1} \cdots (E_2)^{-1} \cdot (E_1)^{-1}$ is zero too, a contradiction.