1111: Linear Algebra I

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Lecture 2

Properties of scalar product

Theorem. For any vectors $\mathbf{v}$, $\mathbf{w}$, $\mathbf{w}_1$, $\mathbf{w}_2$, and any number c, we have

$$\mathbf{v}\cdot\mathbf{w}=\mathbf{w}\cdot\mathbf{v},$$
$$\mathbf{v}\cdot(\mathbf{w}_1+\mathbf{w}_2)=\mathbf{v}\cdot\mathbf{w_1}+\mathbf{v}\cdot\mathbf{w_2},$$
$$\mathbf{v}\cdot(c\mathbf{w})=c(\mathbf{v}\cdot\mathbf{w}),$$
$$\mathbf{v}\cdot\mathbf{v}=|\mathbf{v}|^2.$$

Moreover, the scalar product $\mathbf{v}\cdot\mathbf{w}$ is equal to the product of magnitudes of the vectors times the cosine of the angle ϕ between them:

$$\mathbf{v}\cdot\mathbf{w}=|\mathbf{v}||\mathbf{w}|\cos\phi\ .$$

Properties of scalar product

Proof . All statements except for the last one trivially follow from the definition (plus, in the next-to-last one, the Pythagoras Theorem). Let us prove the last statement. Because of the triangle rule for adding vectors and the Cosine Theorem, we have

$$|\mathbf{v}||\mathbf{w}|\cos\phi = \frac{1}{2}(|\mathbf{v}|^2 + |\mathbf{w}|^2 - |\mathbf{w}-\mathbf{v}|^2) =$$
$$= \frac{1}{2}((a^2+b^2+c^2) + (a_1^2+b_1^2+c_1^2) - (a-a_1)^2 - (b-b_1)^2 - (c-c_1)^2) =$$
$$= aa_1 + bb_1 + cc_1 \ . \quad \square$$

Corollary. For any vectors $\mathbf{v}$ and $\mathbf{w}$, we have

$$-|\mathbf{v}||\mathbf{w}| \leq \mathbf{v}\cdot\mathbf{w} \leq |\mathbf{v}||\mathbf{w}| \ .$$

Note that this statement is trivial from the geometric viewpoint (since $\cos\phi$ is berween -1 and 1), but is harder to infer algebraically.

Vector product

Vector product (cross product): algebraically, if $\mathbf{v}$ has coordinates a, b, c and $\mathbf{w}$ has coordinates a_1, b_1, c_1, then, by definition, $\mathbf{v} \times \mathbf{w}$ has coordinates

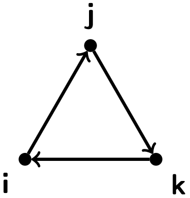
$$bc_1 - b_1c, ca_1 - c_1a, ab_1 - a_1b \ .$$

For example, for the "standard unit vectors" $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, we have

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0,$$
$$\mathbf{i} \times \mathbf{j} = \mathbf{k} = -\mathbf{j} \times \mathbf{i},$$
$$\mathbf{j} \times \mathbf{k} = \mathbf{i} = -\mathbf{k} \times \mathbf{j},$$
$$\mathbf{k} \times \mathbf{i} = \mathbf{j} = -\mathbf{i} \times \mathbf{k}.$$

Vector product

There is a useful mnemonic rule to memorise vector products of standard unit vectors. Draw a triangle with vertices labelled $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, and sides directed $\mathbf{i} \to \mathbf{j} \to \mathbf{k} \to \mathbf{i}$:



Then the vector product of two different standard unit vectors is equal to plus or minus the third one, plus if the order of factors agrees with the direction of the arrow, and minus otherwise, e.g. $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$, because the arrow is directed from $\mathbf{i}$ to $\mathbf{j}$.

Properties of vector product

Theorem. For any vectors $\mathbf{v}$, $\mathbf{w}$, $\mathbf{w}_1$, $\mathbf{w}_2$, and any number c, we have

$$\mathbf{v}\times\mathbf{w} = -\mathbf{w}\times\mathbf{v},$$
$$\mathbf{v}\times(\mathbf{w}_1+\mathbf{w}_2) = \mathbf{v}\times\mathbf{w_1}+\mathbf{v}\times\mathbf{w_2},$$
$$\mathbf{v}\times(c\mathbf{w}) = c(\mathbf{v}\times\mathbf{w}),$$
$$\mathbf{v}\times\mathbf{v} = 0,$$
$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = -\mathbf{v}\cdot(\mathbf{u}\times\mathbf{w}),$$
$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v}-(\mathbf{u}\cdot\mathbf{v})\mathbf{w}.$$

Note that the property $(\mathbf{u}\times\mathbf{v})\times\mathbf{w} = \mathbf{u}\times(\mathbf{v}\times\mathbf{w})$ is not satisfied for vector products, making it very different from products of numbers. The property that does hold instead is the so called "Jacobi identity"

$$(\mathbf{u}\times\mathbf{v})\times\mathbf{w}+(\mathbf{v}\times\mathbf{w})\times\mathbf{u}+(\mathbf{w}\times\mathbf{u})\times\mathbf{v} = 0.$$

Properties of vector product

Proof . All statements except for the last two trivially follow from the definition.
The next-to-last one is already somewhat tedious but not yet a disaster. We have

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = u_1(v_2w_3 - v_3w_2) + u_2(v_3w_1 - v_1w_3) + u_3(v_1w_2 - v_2w_1),$$
$$\mathbf{v}\cdot(\mathbf{u}\times\mathbf{w}) = v_1(u_2w_3 - u_3w_2) + v_2(u_3w_1 - u_1w_3) + v_3(u_1w_2 - u_2w_1),$$

so we can directly see that these two quantities add up to zero.
The last property, meanwhile, is an equation involving some thirty terms, so even though in principle it does follow from the definition, checking it by hand is a "cruel and unusual punishment" (which should be avoided, see, e.g. Eighth Amendment to the United States Constitution).

Therefore, we shall explain how one of the key ideas of linear algebra — that of *linearity* — can be used to simplify the proof dramatically.

(Multi)linearity

Recall that we already know the following: for the given (scalar or vector) product of two vectors $\mathbf{v}$ and $\mathbf{w}$, if replace one of the vectors, say $\mathbf{w}$ by a sum of two vectors, say $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2$, then the result is the same as when you first compute the products with individual summands and then add the results. The same holds for re-scaling: if in a product you multiply one of the vectors by a number, the result is the same as when you compute the product and then multiply the result by that number.

This long paragraph just means that

$$(\mathbf{u}_1 + \mathbf{u}_2) \cdot \mathbf{v} = \mathbf{u}_1 \cdot \mathbf{v} + \mathbf{u}_2 \cdot \mathbf{v},$$

$$\mathbf{u} \cdot (\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{u} \cdot \mathbf{v}_1 + \mathbf{u} \cdot \mathbf{v}_2,$$

$$(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v}),$$

$$(\mathbf{u}_1 + \mathbf{u}_2) \times \mathbf{v} = \mathbf{u}_1 \times \mathbf{v} + \mathbf{u}_2 \times \mathbf{v},$$

$$\mathbf{u} \times (\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{u} \times \mathbf{v}_1 + \mathbf{u} \times \mathbf{v}_2,$$

$$(c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v}) = c(\mathbf{u} \times \mathbf{v}).$$

(Multi)linearity

In general, a function f of several vectors $\mathbf{u}, \mathbf{v}, \ldots$ is said to be *linear in all its arguments* (or *multilinear*) if

$$\begin{gathered}
f(\mathbf{u}_1 + \mathbf{u}_2, \mathbf{v}, \ldots) = f(\mathbf{u}_1, \mathbf{v}, \ldots) + f(\mathbf{u}_2, \mathbf{v}, \ldots),\\
f(c\mathbf{u}, \mathbf{v}, \ldots) = cf(\mathbf{u}, \mathbf{v}, \ldots),\\
f(\mathbf{u}, \mathbf{v}_1 + \mathbf{v}_2, \ldots) = f(\mathbf{u}, \mathbf{v}_1, \ldots) + f(\mathbf{u}, \mathbf{v}_2, \ldots),\\
f(\mathbf{u}, c\mathbf{v}, \ldots) = cf(\mathbf{u}, \mathbf{v}, \ldots),\\
\ldots
\end{gathered}$$

Using dot and cross products, it is easy to build many multilinear functions; for example,

$$\mathbf{u}, \mathbf{v}, \mathbf{w} \mapsto \mathbf{u} \times (\mathbf{v} \times \mathbf{w})$$

is a multilinear function, which immediately follows from the linearity of cross products.

(Multi)linearity

Multilinearity principle. Every multilinear function $f(\mathbf{u}, \mathbf{v}, \ldots)$ is determined by its values assumed when each of the vectors $\mathbf{u}$, $\mathbf{v}$, ... is one of the standard unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$.

Proof. We have

$$f(\mathbf{u}, \mathbf{v}, \ldots) = f(u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}, \mathbf{v}, \ldots) =$$

$$= u_1 f(\mathbf{i}, \mathbf{v}, \ldots) + u_2 f(\mathbf{j}, \mathbf{v}, \ldots) + u_3 f(\mathbf{k}, \mathbf{v}, \ldots) =$$

$$u_1 f(\mathbf{i}, v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}, \ldots)+u_2 f(\mathbf{j}, v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}, \ldots)+u_3 f(\mathbf{k}, v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}, \ldots) =$$

$$u_1 v_1 f(\mathbf{i}, \mathbf{i}, \ldots) + u_1 v_2 f(\mathbf{i}, \mathbf{j}, \ldots) + \ldots = \ldots$$

and continuing this way until all the arguments are standard unit vectors clearly proves the statement. □