1111: Linear Algebra I

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Lecture 20

Linear maps

Example 1. Let V be the vector space of all polynomials in one variable x. Consider the function $\alpha\colon V \to V$ that maps every polynomial $f(x)$ to $3f(x)f'(x)$. This is not a linear map; for example, $1 \mapsto 0$, $x \mapsto 3x$, but $x + 1 \mapsto 3(x + 1) = 3x + 3 \neq 3x + 0$.

Definition 1. Let $\varphi\colon V \to W$ be a linear map, and let $e_1, \ldots, e_n$ and $f_1, \ldots, f_m$ be bases of V and W respectively. Let us compute coordinates of the vectors $\varphi(e_i)$ with respect to the basis $f_1, \ldots, f_m$:

$$\begin{aligned}\varphi(e_1) &= a_{11}f_1 + a_{21}f_2 + \cdots + a_{m1}f_m,\\ \varphi(e_2) &= a_{12}f_1 + a_{22}f_2 + \cdots + a_{m2}f_m,\\ &\ldots\\ \varphi(e_n) &= a_{1n}f_1 + a_{2n}f_2 + \cdots + a_{mn}f_m.\end{aligned}$$

The matrix

$$A_{\varphi,\mathbf{e},\mathbf{f}} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n}\\ a_{21} & a_{22} & \ldots & a_{2n}\\ \vdots & \ldots & \ddots & \vdots\\ a_{m1} & a_{m2} & \ldots & a_{mn}\end{pmatrix}$$

is called *the matrix of the linear map* φ *with respect to the given bases.* For each k, its k-th column is the column of coordinates of image $\varphi(e_k)$.

Similarly to how we proved it for transition matrices, we have the following result.

Lemma 1. *Let* $\varphi\colon V \to W$ *be a linear map, and let* $e_1, \ldots, e_n$ *and* $f_1, \ldots, f_m$ *be bases of* V *and* W *respectively. Suppose that* $x_1, \ldots, x_n$ *are coordinates of some vector* v *relative to the basis* $e_1, \ldots, e_n$*, and* $y_1, \ldots, y_m$ *are coordinates of* $\varphi(v)$ *relative to the basis* $f_1, \ldots, f_m$*. In the notation above, we have*

$$\begin{pmatrix} y_1\\ y_2\\ \vdots\\ y_m\end{pmatrix} = A_{\varphi,\mathbf{e},\mathbf{f}}\begin{pmatrix} x_1\\ x_2\\ \vdots\\ x_n\end{pmatrix}.$$

Proof. The proof is indeed very analogous to the one for transition matrices: we have

$$v = x_1e_1 + \cdots + x_ne_n,$$

so that

$$\varphi(v) = x_1\varphi(e_1) + \cdots + x_n\varphi(e_n).$$

Substituting the expansion of $f(e_i)$'s in terms of f_j's, we get

$$\begin{aligned}\varphi(v) = x_1(a_{11}f_1 + a_{21}f_2 + \cdots + a_{m1}f_m) + \cdots + x_n(a_{1n}f_1 + a_{2n}f_2 + \cdots + a_{mn}f_m) = \\ = (a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n)f_1 + \cdots + (a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n)f_n.\end{aligned}$$

Since we know that coordinates are uniquely defined, we conclude that

$$\begin{gathered}a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = y_1, \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = y_n,\end{gathered}$$

which is what we want to prove. □

The next statement is also similar to the corresponding one for transition matrices (also, we sort of used this statement implicitly when we talked about matrix products long ago).

Lemma 2. *Let U, V, and W be vector spaces, and let $\psi\colon U \to V$ and $\varphi\colon V \to W$ be linear maps. Finally, let $e_1, \ldots, e_n$, $f_1, \ldots, f_m$, and $g_1, \ldots, g_k$ be bases of U, V, and W respectively. Then*

$$A_{\varphi\circ\psi,\mathbf{e},\mathbf{g}} = A_{\varphi,\mathbf{f},\mathbf{g}}A_{\psi,\mathbf{e},\mathbf{f}}.$$

Proof. First, let us note that

$$\begin{gathered}(\varphi\circ\psi)(u_1+u_2) = \varphi(\psi(u_1+u_2)) = \varphi(\psi(u_1)+\psi(u_2)) = \varphi(\psi(u_1)) + \varphi(\psi(u_2)) = (\varphi\circ\psi)(u_1) + (\varphi\circ\psi)(u_2), \\ (\varphi\circ\psi)(c\cdot u) = \varphi(\psi(c\cdot u)) = \varphi(c\psi(u)) = c\varphi(\psi(u)) = c(\varphi\circ\psi)(u),\end{gathered}$$

so $\varphi\circ\psi$ is a linear map.

Let us prove the second statement. We take a vector $\mathbf{u} \in U$, and apply the formula of Lemma 2. On the one hand, we have

$$(\varphi\circ\psi(\mathbf{u}))_{\mathbf{g}} = A_{\varphi\circ\psi,\mathbf{e},\mathbf{g}}\mathbf{u}_{\mathbf{e}}.$$

On the other hand, we obtain,

$$(\varphi\circ\psi(\mathbf{u}))_{\mathbf{g}} = (\varphi(\psi(\mathbf{u})))_{\mathbf{g}} = A_{\varphi,\mathbf{f},\mathbf{g}}(\psi(\mathbf{u})_{\mathbf{f}}) = A_{\varphi,\mathbf{f},\mathbf{g}}(A_{\psi,\mathbf{e},\mathbf{f}}\mathbf{u}_{\mathbf{e}}) = (A_{\varphi,\mathbf{f},\mathbf{g}}A_{\psi,\mathbf{e},\mathbf{f}})\mathbf{u}_{\mathbf{e}}.$$

Therefore

$$A_{\varphi\circ\psi,\mathbf{e},\mathbf{g}}\mathbf{u}_{\mathbf{e}} = (A_{\varphi,\mathbf{f},\mathbf{g}}A_{\psi,\mathbf{e},\mathbf{f}})\mathbf{u}_{\mathbf{e}}$$

for every $u_{\mathbf{e}}$. From our previous classes we know that knowing $A\mathbf{v}$ for all vectors $\mathbf{v}$ completely determines the matrix A, so

$$A_{\varphi\circ\psi,\mathbf{e},\mathbf{g}} = A_{\varphi,\mathbf{f},\mathbf{g}}A_{\psi,\mathbf{e},\mathbf{f}},$$

as required. □

Example 2. Let us consider the linear map $X\colon P_2 \to P_3$ discussed earlier. Let us take the bases $e_1 = 1, e_2 = x, e_3 = x^2$ of P_2, and the basis $f_1 = 1, f_2 = x, f_3 = x^2, f_4 = x^3$ of P_3, and compute $A_{X,\mathbf{e},\mathbf{f}}$. Note that $X(e_1) = x\cdot 1 = x = f_2$, $X(e_2) = x\cdot x = x^2 = f_3$, and $X(e_3) = x\cdot x^2 = x^3 = f_4$. Therefore

$$A_{X,\mathbf{e},\mathbf{f}} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Example 3. Let us consider the linear map $D\colon P_3 \to P_2$. Let us take the bases $e_1 = 1, e_2 = x, e_3 = x^2, e_4 = x^3$ of P_3, and the basis $f_1 = 1, f_2 = x, f_3 = x^2$ of P_2, and let us compute $A_{D,\mathbf{e},\mathbf{f}}$. Note that $D(e_1) = 1' = 0$, $D(e_2) = x' = 1 = f_1$, $D(e_3) = (x^2)' = 2x = 2f_2$, and $D(e_4) = (x^3)' = 3x^2 = 3f_3$. Therefore

$$A_{D,\mathbf{e},\mathbf{f}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Example 4. It is also useful to remark that $A_{D,\mathbf{e},\mathbf{f}} A_{X,\mathbf{e},\mathbf{f}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, which is indeed $A_{D\circ C,\mathbf{e},\mathbf{e}}$ as our previous lemma suggests.

Example 5. Consider the vector space M_2 of all 2×2-matrices. Let us define a function $\beta\colon M_2 \to M_2$ by the formula $\beta(X) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} X$. Let us check that this map is a linear operator. Indeed, by properties of matrix products

$$\beta(X_1 + X_2) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} (X_1 + X_2) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} X_1 + \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} X_2 = \beta(X_1) + \beta(X_2),$$
$$\beta(cX) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} (cX) = c \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} X = c\beta(X).$$

We consider the basis of matrix units in M_2: $e_1 = E_{11}$, $e_2 = E_{12}$, $e_3 = E_{21}$, $e_4 = E_{22}$. We have

$$\beta(e_1) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = e_1 + e_3,$$
$$\beta(e_2) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} e_2 + e_4,$$
$$\beta(e_3) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = e_1,$$
$$\beta(e_4) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = e_2,$$

so

$$A_{\beta,\mathbf{e}} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

It is important to remark that a lot of students get this last example wrong, saying that the matrix of β is $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ since β multiplies every matrix by $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. The problem here is that the space of 2×2-matrices is four-dimensional so β must be represented by a 4×4-matrix.