1111: Linear Algebra I

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Lecture 1

House rules

- 3 classes every week (Thursday 9am MacNeil, Friday 9am LB04, Friday 1pm PLLT)
- All lectures on odd weeks, two lectures and a tutorial on even weeks. Tutorial work does not affect your mark, but prepares your for homework and exam. Plus, you can ask questions in tutorials, and get some hints from which you learn. (You can even work in groups in tutorials if you can do it quietly without disturbing others).
- Homeworks are due after our 1pm class every Friday from the second week onwards. Later the same day solutions will be posted, so no late homeworks will be accepted!

Structure of exposition

In most maths modules, including this one, there will be several types of material presented:

- **Definitions**: notions being introduced for the first time;
- **Methods**: steps one is recommended to undertake in order to solve a particular type of questions;
- **Theorems**: theoretical statements upon which methods rely, directly or indirectly, and which are used to establish properties of notions that we defined. Some auxiliary theoretical statements are called *lemmas*, some less central results are called *propositions*, and finally some results that are in some way direct consequences of other results proved are called *corollaries*. Effectively, all these are "theorems of different magnitude".
- **Proofs**: Mathematically rigorous reasonings that justify statements of theorems.

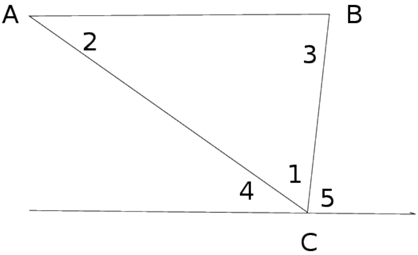
Example: sum of angles in a triangle

We shall discuss two proofs of a very well known result from planar geometry:

Theorem. In every triangle, the sum of the angles is equal to the straight angle (180°).

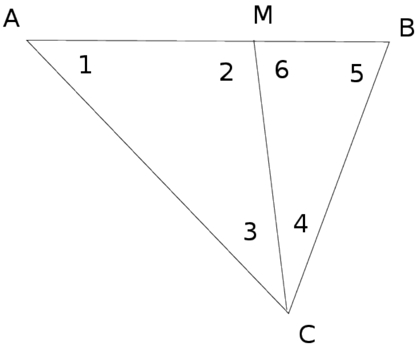
There is a catch though. One of the proofs is correct, the other one contains a mistake — your task is to decide which is which.

First proof



$$\angle 1 + \angle 2 + \angle 3 = \angle 1 + \angle 4 + \angle 5 = 180^\circ.$$

Second proof

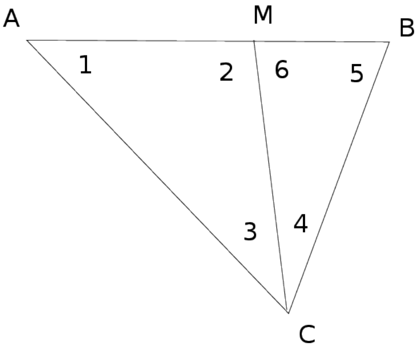


$$S + S = (\angle 1 + \angle 2 + \angle 3) + (\angle 4 + \angle 5 + \angle 6) =$$
$$= (\angle 1 + (\angle 3 + \angle 4) + \angle 5) + (\angle 2 + \angle 6) = S + 180^\circ,$$

so $S = 180^\circ$.

Mystery revealed

In fact, the second proof does not prove the theorem:



we may only conclude that

$$S_1 + S_2 = (\angle 1 + \angle 2 + \angle 3) + (\angle 4 + \angle 5 + \angle 6) =$$
$$= (\angle 1 + (\angle 3 + \angle 4) + \angle 5) + (\angle 2 + \angle 6) = S + 180^\circ \ !$$

"Good news"

However, that "proof" was not entirely pointless! The property

$$S_1 + S_2 = S + 180^\circ$$

can be rewritten as

$$(S_1 - 180^\circ) + (S_2 - 180^\circ) = S - 180^\circ,$$

so the quantity $S - 180^\circ$, also called the *defect*, is "additive": for a triangle glued from smaller ones, its defect is equal to the sum of defects of parts.

In fact, there exist "non-Euclidean geometries" for which a triangle may have nonzero defect.

What is linear algebra?

Informally, *linear algebra* tells you how to use *algebra* in order to deal with *linear* (flat) geometric objects (lines, planes, vectors, and so on).
For example, let us consider the function

$$F(x, y, z) = x^2 + 2xy - 3yz + y^2 - 34z^2.$$

What is the maximal and the minimal value of this function if the point (x, y, z) belongs to the sphere $x^2 + y^2 + z^2 = 1$? Linear algebra provides easy methods that allow to answer this question.
Generally, applications of maths often need to deal with curved objects of complicated shapes. However, it is often possible to re-formulate problems in such a way that one only cares about the "local" behaviour of those objects, e.g. close to some given point M. Then, *analysis* provides one with methods to reduce such questions to questions about flat objects, and *linear algebra* solves those questions. This way, linear algebra is "analysis done with a microscope".

Recollections on vectors in 2D and 3D

The central notion of linear algebra is that of a vector. Geometrically, vector is a directed segment between two points. Two vectors are said to be equal if they have the same magnitude (length) and direction.
Algebraically, a vector is defined by its coordinates: a pair of numbers for 2D vectors, and a triple for 3D vectors.
There are several standard operations on vectors: the most notable are addition, re-scaling, scalar product (dot product), and, in 3D, vector product (cross product).

Addition: the vector $\mathbf{v} + \mathbf{w}$ can be defined geometrically ("triangle rule" or "parallelogram rule") or algebraically (adding respective coordinates)

Re-scaling: geometrically, if c is a number, then $c\mathbf{v}$ is the vector whose magnitude is $|c|$ times the magnitude of $\mathbf{w}$, and direction the same as the direction of $\mathbf{v}$ for $c > 0$ and the opposite one for $c < 0$. (It is the *null* vector $\mathbf{0}$ for $c = 0$ or $\mathbf{v} = 0$).

Scalar product

Scalar product (dot product): algebraically, one adds the produts of respective coordinates, e.g. in 3D, if **v** has coordinates a, b, c and **w** has coordinates a_1, b_1, c_1, then

$$\mathbf{v} \cdot \mathbf{w} = aa_1 + bb_1 + cc_1 \ ,$$

geometrically we shall define it a bit later. The result is a scalar (number), hence the name *scalar product*.

Here and below, the respective formulas in 2D are obtained by associating to a 2D vector with coordinates a, b the 3D vector with coordinates $a, b, 0$. I shall not emphasize that every time.

Properties of scalar product

Theorem. For any vectors $\mathbf{v}$, $\mathbf{w}$, $\mathbf{w}_1$, $\mathbf{w}_2$, and any number c, we have

$$\mathbf{v}\cdot\mathbf{w}=\mathbf{w}\cdot\mathbf{v},$$
$$\mathbf{v}\cdot(\mathbf{w}_1+\mathbf{w}_2)=\mathbf{v}\cdot\mathbf{w_1}+\mathbf{v}\cdot\mathbf{w_2},$$
$$\mathbf{v}\cdot(c\mathbf{w})=c(\mathbf{v}\cdot\mathbf{w}),$$
$$\mathbf{v}\cdot\mathbf{v}=|\mathbf{v}|^2.$$

Moreover, the scalar product $\mathbf{v}\cdot\mathbf{w}$ is equal to the product of magnitudes of the vectors times the cosine of the angle ϕ between them:

$$\mathbf{v}\cdot\mathbf{w}=|\mathbf{v}||\mathbf{w}|\cos\phi\ .$$