1111: Linear Algebra I

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Lecture 18

Dimension

Note that in $\mathbb{R}^n$ we proved that a linearly independent system of vectors consists of at most n vectors, and a complete system of vectors consists of at least n vectors. In a general vector space V, there is no *a priori* n that can play this role. Moreover, the previous example shows that sometimes, no n bounding the size of a linearly independent system of vectors may exist. It however is possible to prove a version of those statements which is valid in every vector space.

Theorem 1. *Let V be a vector space, and suppose that $e_1, \ldots, e_k$ is a linearly independent system of vectors and that $f_1, \ldots, f_m$ is a complete system of vectors. Then $k \leqslant m$.*

Proof. Assume the contrary; without loss of generality, $k > m$. Since $f_1, \ldots, f_m$ is a complete system, we can find coefficients a_{ij} for which

$$\begin{aligned} e_1 &= a_{11}f_1 + a_{21}f_2 + \cdots + a_{m1}f_m, \\ e_2 &= a_{12}f_1 + a_{22}f_2 + \cdots + a_{m2}f_m, \\ &\ldots \\ e_k &= a_{1k}f_1 + a_{2k}f_2 + \cdots + a_{mk}f_m. \end{aligned}$$

Let us look for linear combinations $c_1e_1 + \cdots + c_kv_k$ that are equal to zero (since these vectors are assumed linearly independent, we should not find any nontrivial ones). Such a combination, once we substitute the expressions above, becomes

$$c_1(a_{11}f_1+a_{21}f_2+\cdots+a_{m1}f_m)+c_2(a_{12}f_1+a_{22}f_2+\cdots+a_{m2}f_m)+\ldots+c_k(a_{1k}f_1+a_{2k}f_2+\cdots+a_{mk}f_m) = \\ = (a_{11}c_1 + a_{12}c_2 + \cdots + a_{1k}c_k)f_1 + \cdots + (a_{m1}c_1 + a_{m2}c_2 + \cdots + a_{mk}c_k)f_m.$$

This means that if we ensure

$$\begin{aligned} a_{11}c_1 + a_{12}c_2 + \cdots + a_{1k}c_k &= 0, \\ &\ldots \\ a_{m1}c_1 + a_{m2}c_2 + \cdots + a_{mk}c_k &= 0, \end{aligned}$$

then this linear combination is automatically zero. But since we assume $k > m$, this system of linear equations has a nontrivial solution $c_1, \ldots, c_k$, so the vectors $e_1, \ldots, e_k$ are linearly dependent, a contradiction. □

This result leads, indirectly, to an important new notion.

Definition 1. We say that a vector space V is *finite-dimensional* if it has a basis consisting of finitely many vectors. Otherwise we say that V is *infinite-dimensional.*

Example 1. Clearly, $\mathbb{R}^n$ is finite-dimensional. The space of all polynomials is infinite-dimensional: finitely many polynomials can only produce polynomials of bounded degree as linear combinations.

Lemma 1. *Let V be a finite-dimensional vector space. Then every basis of V consists of the same number of vectors.*

Proof. Indeed, having a basis consisting of n elements implies, in particularly, having a complete system of n vectors, so by our theorem, it is impossible to have a linearly independent system of more than n vectors. Thus, every basis has finitely many elements, and for two bases $e_1, \ldots, e_k$ and $f_1, \ldots, f_m$ we have $k \leqslant m$ and $m \leqslant k$, so $m = k$. □

Definition 2. For a finite-dimensjonal vector V, the number of vectors in a basis of V is called the *dimension* of V, and is denoted by $\dim(V)$.

Example 2. The dimension of $\mathbb{R}^n$ is equal to n, as expected.

Example 3. The dimension of the space of polynomials in one variable x of degree at most n is equal to $n+1$, since it has a basis $1, x, \ldots, x^n$.

Example 4. The dimension of the space of $m \times n$-matrices is equal to mn.

Coordinates

Let V be a finite-dimensional vector space, and let $e_1, \ldots, e_n$ be a basis of V.

Definition 3. For a vector $v \in V$, the scalars $c_1, \ldots, c_n$ for which

$$v = c_1 e_1 + c_2 e_2 + \cdots + c_n e_n$$

are called the *coordinates of v relative to the basis* $e_1, \ldots, e_n$.

Lemma 2. *The above definition makes sense: each vector has (unique) coordinates.*

Proof. Existence follows from the spanning property of a basis, uniqueness — from the linear independence. □

If v has coordinates $c_1, c_2, \ldots, c_n$ and w has coordinates $d_1, d_2, \ldots, d_n$ (relative to the same basis!), then $v+w$ has coordinates $c_1+d_1, c_2+d_2, \ldots, c_n+d_n$, and for any scalar c, the vector $c \cdot v$ has coordinates $cc_1, cc_2, \ldots, cc_n$. Therefore, choosing a basis effectively identifies V with $\mathbb{R}^n$. However, choosing a convenient basis might simplify computations drastically, and that is where methods of linear algebra are particularly beneficial.

Change of coordinates

Let V be a vector space of dimension n, and let $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ be two different bases of V.

Definition 4. Let us express the vectors $f_1, \ldots, f_n$ as linear combinations of $e_1, \ldots, e_n$:

$$\begin{aligned} f_1 &= a_{11}e_1 + a_{21}e_2 + \cdots + a_{m1}e_m, \\ f_2 &= a_{12}e_1 + a_{22}e_2 + \cdots + a_{m2}e_m, \\ &\cdots \\ f_n &= a_{1n}e_1 + a_{2n}e_2 + \cdots + a_{mn}e_m. \end{aligned}$$

The matrix (a_{ij}) is called *the transition matrix* from the basis $e_1, \ldots, e_n$ to the basis $f_1, \ldots, f_n$. Its k-th column is the column of coordinates of the vector f_k relative to the basis $e_1, \ldots, e_n$.