1111: Linear Algebra I

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Lecture 16

Spaces of polynomials

Let us mention another source of examples of vector spaces which we shall be using quite frequently.

Example 1. The set of all polynomials

$$a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0$$

in one variable x with real coefficients is a vector space with respect to addition and re-scaling. If we consider polynomials of degree at most n for some given n, this is also a vector space. Polynomials of degree exactly n do not form a vector space, since the sum of two such polynomials may have smaller degree, e.g. $x^n + (1 - x^n) = 1$, where the sum of two polynomials of degree n is of degree 0. (Also, polynomials of degree exactly n do not contain the zero polynomial.)

Definition 1. A subset U of a vector space V is called a subspace if it contains 0, is closed under addition and multiplication by scalars, and is closed under taking opposites.

As in the case of $\mathbb{R}^n$, a subspace of a vector space V (with the inherited operations of addition and multiplication by scalars) is again a vector space: all the properties are trivially satisfied.

Example 2. The set of all polynomials $f(x)$ such that $f(0) = 0$ is a subspace, since it contains the zero polynomial, and closed under addition, multiplication by scalars, and taking opposites.

The set of all polynomials $f(x)$ such that $f(0) = 1$ is not a subspace (none of the conditions are satisfied).

The set of all polynomials with integer coefficients is not a subspace, since it is not closed under multiplication by scalars.

The set of all polynomials with at most one nonzero coefficient is closed under multiplication by scalars, but not under addition.

In the next home assignment, you will consider some further subspaces of the space of polynomials.

Fields

It is also worth mentioning that sometimes we shall use other scalars, not just real numbers. In order for all the arguments to work, we need that scalars have arithmetics similar to that of real numbers. Let us be precise about what that means.

Definition 2. A *field* is a set F equipped with the following data:

- a rule assigning to each elements $f_1, f_2 \in F$ an element of F denoted $v_1 + v_2$, and
- a rule assigning to each elements $f_1, f_2 \in F$ an element of F denoted $f_1 \cdot f_2$ (or sometimes $f_1 f_2$),

for which the following properties are satisfied:

1. for all $f_1, f_2, f_3 \in F$ we have $(f_1 + f_2) + f_3 = f_1 + (f_2 + f_3)$,

2. for all $f_1, f_2 \in F$ we have $f_1 + f_2 = f_2 + f_1$,
3. there is a designated element of F denoted by 0 for which $f + 0 = f$ for all f,
4. for each $f \in F$, there is a designated element of F denoted $-f$ and called *the opposite of* f such that $f + (-f) = 0$,
5. for all $f_1, f_2, f_3 \in F$ we have $(f_1 f_2) f_3 = f_1 (f_2 f_3)$,
6. for all $f_1, f_2 \in F$ we have $f_1 f_2 = f_2 f_1$,
7. there is a designated element of F denoted by 1 for which $1 \cdot f = f \cdot 1 = f$ for all f,
8. for each $f \neq 0 \in F$, there is a designated element of F denoted f^{-1} and called *the inverse of* f such that $ff^{-1} = 1$,
9. for all $f_1, f_2, f_3 \in F$, we have $f_1 \cdot (f_2 + f_3) = f_1 \cdot f_2 + f_1 \cdot f_3$.

The existence of multiplicative inverses is perhaps the most important and intricate feature of fields. For purposes of linear algebra, that property is absolutely crucial, for that is how one creates pivots in reduced row echelon matrices, making systems of linear equations manageable.

Example 3. The field of real numbers $\mathbb{R}$ is our main example of a field; I assume that you know what it stands for.

Example 4. The field of complex numbers $\mathbb{C}$ consists, as you know, of expressions $a + bi$, where $a, b \in \mathbb{R}$ with obvious addition and multiplication that is completely defined by the rule $i^2 = -1$.

Example 5. The field of rational numbers $\mathbb{Q}$ consists of fractions with integer numerator and integer nonzero denominator (like $1/2$, $-5/3$, etc.).

Example 6. An example which is absolutely foundational for computer science is the binary arithmetic: $\mathbb{F}_2 = \{0, 1\}$ with the operations $0 + 0 = 1 + 1 = 0$, $0 + 1 = 1 + 0 = 1$, $0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0$, $1 \cdot 1 = 1$.

Given a field F, one can consider vector spaces over F, that is vector spaces where elements of F play the role of scalars. The flexibility of choosing scalars for the vector space can sometimes be very useful.

Coin weighing problem

This example is a question where the viewpoint of linear algebra, as well as using different scalars in vector spaces, turns out to be very important. In fact, I am not aware of more “elementary” solutions.

> Given 101 coins of various shapes and denominations, one knows that if you remove any one coin, the remaining 100 coins can be divided into two groups of 50 of equal total weight. Show that all the coins are of the same weight.