1111: Linear Algebra I

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Lecture 14

Previously on...

The vectors $v_1, \ldots, v_k$ are said to be *linearly independent* if the only linear combination of this vector which is equal to the zero vector is the combination where all coefficients are equal to 0. Otherwise those vectors are said to be *linearly dependent*.

The vectors $v_1, \ldots, v_k$ are said to *span* $\mathbb{R}^n$, or to form a *complete set of vectors*, if every vector can be written as some linear combination of those vectors.

We say that vectors $v_1, \ldots, v_k$ in $\mathbb{R}^n$ form a *basis* if they are linearly independent and they span $\mathbb{R}^n$.

Previously on...

Let $v_1, \ldots, v_k$ be vectors in $\mathbb{R}^n$. Consider the $n \times k$-matrix A whose columns are these vectors.

These vectors are linearly independent if and only if the reduced row echelon form of A has a pivot in every column.

These vectors span $\mathbb{R}^n$ if and only if the reduced row echelon form of A has a pivot in every row.

These vectors form a basis if and only if the reduced row echelon form of A is I_n.

Coordinates

Let $e_1, \ldots, e_n$ be a basis of V. For a vector v, the scalars $c_1, \ldots, c_n$ for which

$$v = c_1 e_1 + c_2 e_2 + \cdots + c_n e_n$$

are called the *coordinates of v with respect to the basis* $e_1, \ldots, e_n$.

This definition makes sense: each vector has (unique) coordinates. Existence follows from the spanning property of a basis, uniqueness — from the linear independence.

Let us take the vectors $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, as the last time. These vectors form a basis of $\mathbb{R}^2$. The coordinates of the vector $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with respect to this basis are given by the column $v_e = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, because $\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Subspaces of $\mathbb{R}^n$

A non-empty subset U of $\mathbb{R}^n$ is called a *subspace* if the following properties are satisfied:

- whenever $v, w \in U$, we have $v + w \in U$;
- whenever $v \in U$, we have $c \cdot v \in U$ for every scalar c.

Of course, this implies that every linear combination of several vectors in U is again in U.

Exercise: show that the zero vector is contained in any subspace.

Let us give some examples. Of course, there are two very trivial examples: $U = \mathbb{R}^n$ and $U = \{0\}$.

Example 1: The line $y = x$ in $\mathbb{R}^2$ is another example, since our basic operations can only create vectors with equal coordinates.

Example 2: Any line or 2D plane containing the origin in $\mathbb{R}^3$ would also give an example, and these give a general intuition of what the word "subspace" should make one think of.

Subspaces of $\mathbb{R}^n$

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- whenever $v \in U$, we have $c \cdot v \in U$ for every scalar c.

Non-example 1: Consider all vectors $v = \begin{pmatrix} x \\ y \end{pmatrix}$ in $\mathbb{R}^2$ for which $x = y = 0$ or $x \neq y$. The second property is satisfied but the first one fails since $\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Non-example 2: Consider all vectors with both integer coordinates in $\mathbb{R}^2$. The first property is satisfied, but the second one fails since $\frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$.

Subspaces of $\mathbb{R}^n$: two main examples

Let A be an $m \times n$-matrix. Then the solution set to the homogeneous system of linear equations $Ax = 0$ is a subspace of $\mathbb{R}^n$. Indeed, it is non-empty because it contains $x = 0$. We also see that if $Av = 0$ and $Aw = 0$, then $A(v + w) = Av + Aw = 0$, and similarly if $Av = 0$, then $A(c \cdot v) = c \cdot Av = 0$.

Let $v_1, \ldots, v_k$ be some given vectors in $\mathbb{R}^n$. Their linear span $\text{span}(v_1, \ldots, v_k)$ is the set of all possible linear combinations $c_1 v_1 + \ldots + c_k v_k$. The linear span of $k \geq 1$ vectors is a subspace of $\mathbb{R}^n$. Indeed, it is manifestly non-empty (contains the zero vector), and closed under sums and scalar multiples.

The example of the line $y = x$ from the previous slide fits into both contexts. First of all, it is the solution set to the system of equations $A\mathbf{x} = 0$, where $A = \begin{pmatrix} 1 & -1 \end{pmatrix}$, and $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$. Second, it is the linear span of the vector $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. We shall see that it is a general phenomenon: these two descriptions are equivalent.

Subspaces of $\mathbb{R}^n$: two main examples

Consider the matrix $A = \begin{pmatrix} 1 & -2 & 1 & 0 \\ 3 & -5 & 3 & -1 \end{pmatrix}$, and the system of equations $Ax = 0$. The reduced row echelon form of this matrix is $\begin{pmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 0 & -1 \end{pmatrix}$, so the free unknowns are x_3 and x_4. Setting $x_3 = s$, $x_4 = t$, we obtain the solution $\begin{pmatrix} -s+2t \\ t \\ s \\ t \end{pmatrix}$, which we can represent as $s\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix}$. We conclude that the solution set to the system of equations is the linear span of the vectors $v_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix}$.

Subspaces of $\mathbb{R}^n$: two main examples

Let us implement this approach in general. Suppose A is an $m \times n$-matrix.

As we know, to describe the solution set for $Ax = 0$ we bring A to its reduced row echelon form, and use free unknowns as parameters. Let x_{i_1}, ..., x_{i_k} be free unknowns. For each $j = 1, \ldots, k$, let us define the vector v_j to be the solution obtained by putting the j-th free unknown to be equal to 1, and all others to be equal to zero.

Note that the solution that corresponds to arbitrary values $x_{i_1} = t_1$, ..., $x_{i_k} = t_k$ is the linear combination $t_1 v_1 + \cdots + t_k v_k$. Therefore the solution set of $Ax = 0$ is the linear span of v_1, ..., v_k.

Subspaces of $\mathbb{R}^n$: two main examples

In fact the solution vectors $v_1, \ldots, v_k$ we just constructed linearly independent.

Indeed, the linear combination $t_1 v_1 + \cdots + t_k v_k$ has t_i in the place of i-th free unknown, so if this combination is equal to zero, then all coefficients must be equal to zero.

All in all, it is sensible to say that these vectors form a basis in the subspace of solutions: every vector can be obtained as their linear combination, and they are linearly independent.

However, we only considered bases of $\mathbb{R}^n$ so far, and the solution set of a system of linear equations differs from $\mathbb{R}^m$. After the reading week, we shall rectify that and talk about arbitrary "abstract" vector spaces.