1111: Linear Algebra I

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Lecture 10

Previously on...

Multilinearity is obvious: each of the terms in

$$\det(A) = \sum_{\sigma} \mathrm{sign}(\sigma) A_{i_1 j_1} A_{i_2 j_2} \cdots A_{i_n j_n}$$

contains exactly one term from each row, so if for two matrices all rows but one are the same, in the sum of their determinants we can collect the similar terms, and get the determinant where two rows are added. A similar but easier argument works for scalar multiples.

Change of sign under swapping rows is also clear: swapping rows corresponds to swapping two elements in the top row of the two-row notation of each permutation, so changes the sign.

These two properties together imply that when combining rows, we have $\det(A') = \det(A) + c\det(\tilde{A})$ where $\tilde{A}$ has two equal rows. But then of course $\det(\tilde{A}) = -\det(\tilde{A})$, so $\det(\tilde{A}) = 0$.

Determinants and invertibility

Now we shall see what *determinants* actually *determine*:

> *An $n \times n$-matrix A is invertible if and only if* $\det(A) \neq 0$.

Indeed, let us consider a sequence of elementary row operations that bring A to its reduced row echelon form R. Each of them multiplies $\det(A)$ by a non-zero scalar, so $\det(A) \neq 0$ if and only $\det(R) \neq 0$. It remains to notice that for an invertible matrix A, we have $R = I_n$ with the determinant $1 \neq 0$, and for a matrix which is not invertible, R has a row of zeros, so $\det(R) = 0$, and $\det(A) = 0$.

Properties of determinants

A somewhat nontrivial property of determinants which is almost immediate from our work is

$$\det(A \cdot B) = \det(A)\det(B).$$

Indeed, already for $n = 10$ this amounts to check that in more than 10^{16} terms on the left many do cancel each other, producing some 10^{13} terms on the right.

First, note that we already know it in the case when the matrix A is elementary. Indeed, $\det(EB) = \det(E)\det(B)$ for an elementary matrix E, because multiplying by E corresponds to performing a row operation on B, and we know how determinants behave with respect to row operations.

Hence, up to taking into account common factors $\det(E_i)$, proving our statement is reduced to the case when $A = R$ is in reduced row echelon form. In this case, the proof is easy: either $R = I_n$, so the formula becomes $\det(B) = 1 \cdot \det(B)$, otherwise both R and RB have a row of zeros which makes their determinants equal to zero.

Properties of determinants

It turns out that we can also simplify determinants by performing elementary column operations (same as with rows, but using columns).

The easiest way to justify it utilises the notion of the transpose matrix. For an $m \times n$-matrix A, its transpose A^T is the $n \times m$-matrix whose rows are columns of A. For example,

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 0 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} .$$

Exercise: Show that $(A^T)^T = A$ and $(A \cdot B)^T = B^T \cdot A^T$ whenever the product $A \cdot B$ is defined. Show also that if a square matrix A is invertible, then the matrix A^T is also invertible, and that $(A^T)^{-1} = (A^{-1})^T$.

Properties of determinants

Let us show that $\det(A^T)=\det(A)$. Indeed, note that in the formula

$$\det(A)=\sum_{\sigma}\operatorname{sign}(\sigma)A_{i_1j_1}A_{i_2j_2}\cdots A_{i_nj_n}$$

passing from A to A^T amounts to swapping i_k with j_k, that is swapping the first and the second row of the two-row matrix representing the permutation. As we know, it does not affect the sign of the permutation, since to compute it we count the total number of inversions in two rows. $\square$

Of course, this property implies that elementary column operations have the same effect on determinants as respective elementary row operations. This can sometimes be useful in computations.

Minors and cofactors

Our next goal will be to prove some formulas involving determinants. Even though we already know enough to compute determinants in practice, in some cases existence of formulas of a particular shape is important for applications. (An idle example: suppose that each element of a $10^4 \times 10^4$-matrix A is a real number between -100 and 100, and we know these numbers with precision 10^{-5}. With what precision can we know the determinant of A?)

For an $n \times n$-matrix A with entries a_{ij}, we denote by A^{ij} the i, j-*minor* of A, that is the determinant of the $(n-1) \times (n-1)$-matrix obtained from A by removing the i-th row and the j-th column. For example, let us consider the matrix $A = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 1 & -2 \\ 0 & 1 & 1 \end{pmatrix}$. Then $A^{11} = 3$, $A^{12} = 2$, $A^{13} = 2$, $A^{21} = 3$, $A^{22} = 1$, $A^{23} = 1$, $A^{31} = -6$, $A^{32} = -2$, $A^{33} = -5$.

Minors and cofactors

A notion that simplifies many formulas is that of a *cofactor*. Cofactors are "minors with signs": for the given matrix A, its cofactors C^{ij} are defined by the formula $C^{ij} = (-1)^{i+j}A^{ij}$.

In simple words, the extra signs follow a chessboard pattern: we put $+1$ in the top left corner, and then keep changing signs once we cross borders between rows / columns.

For example, for the matrix $A = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 1 & -2 \\ 0 & 1 & 1 \end{pmatrix}$ we already computed $A^{11} = 3$, $A^{12} = 2$, $A^{13} = 2$, $A^{21} = 3$, $A^{22} = 1$, $A^{23} = 1$, $A^{31} = -6$, $A^{32} = -2$, $A^{33} = -5$, so we have $C^{11} = 3$, $C^{12} = -2$, $C^{13} = 2$, $C^{21} = -3$, $C^{22} = 1$, $C^{23} = -1$, $C^{31} = -6$, $C^{32} = 2$, $C^{33} = -5$.

Row expansion of the determinant

Our next goal is to prove the following formula:

$$\det(A) = a_{i1}C^{i1} + a_{i2}C^{i2} + \cdots + a_{in}C^{in},$$

in other words, if we multiply each element of the i-th row of A by its cofactor and add results, the number we obtain is equal to $\det(A)$.

Let us first handle the case $i = 1$. In this case, once we write the corresponding minors with signs, the formula reads

$$\det(A) = a_{11}A^{11} - a_{12}A^{12} + \cdots + (-1)^{n+1}a_{1n}A^{1n}.$$

Let us examine the formula for $\det(A)$. It is a sum of terms corresponding to pick n elements representing each row and each column of A. In particular, it involves picking an element from the first row. This can be one of the elements $a_{11}, a_{12}, \ldots, a_{1n}$. What remains, after we picked the element a_{1k}, is to pick other elements; now we have to avoid the row 1 and the column k. This means that the elements we need to pick are precisely those involved in the determinant A^{1k}, and we just need to check that the signs match.