MA 1111: Linear Algebra I

Homework problems for November 23, 2018

Solutions to this problem sheet are to be handed in after our class at 1pm on Friday. Please attach a cover sheet with a declaration `http://tcd-ie.libguides.com/plagiarism/declaration` confirming that you know and understand College rules on plagiarism. On the same cover sheet, please put your name, student number, and name of the degree (Maths/TP/TSM), and staple all the sheets together. (Failure to do that may result in misplaced/lost sheets, for which no responsibility can be taken by instructors.)

1. The (counterclockwise) rotation through $90°$ around the origin is a linear operator from $\mathbb{R}^2$ to $\mathbb{R}^2$. Write down its matrix relative to the basis (**a**) $\begin{pmatrix}1\\0\end{pmatrix}$, $\begin{pmatrix}0\\1\end{pmatrix}$; (**b**) $\begin{pmatrix}1\\0\end{pmatrix}$, $\begin{pmatrix}1\\1\end{pmatrix}$.

2. The matrix of a linear operator $\mathcal{A}\colon \mathbb{R}^3 \to \mathbb{R}^3$ is (relative to the basis of standard unit vectors) $\begin{pmatrix}1 & 1 & 0\\0 & 1 & 1\\0 & 0 & 1\end{pmatrix}$. Compute its matrix in the basis (**a**) $\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\begin{pmatrix}1\\1\\0\end{pmatrix}$, $\begin{pmatrix}0\\2\\1\end{pmatrix}$; (**b**) $\begin{pmatrix}1\\-1\\1\end{pmatrix}$, $\begin{pmatrix}0\\1\\0\end{pmatrix}$, $\begin{pmatrix}1\\1\\0\end{pmatrix}$.

3. A sequence $b_0, b_1, \ldots$ is defined by $b_0 = 0$, $b_1 = 1$, $b_{n+1} = 3b_n - b_{n-1}$.

(**a**) Show that that $\begin{pmatrix}0 & 1\\-1 & 3\end{pmatrix}^n \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}b_n\\b_{n+1}\end{pmatrix}$.

(**b**) Find eigenvalues and eigenvectors of $\begin{pmatrix}0 & 1\\-1 & 3\end{pmatrix}$ and use them to obtain an explicit formula for b_n.

4. Determine eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix}2 & -1 & 1\\0 & 1 & 1\\1 & -1 & 1\end{pmatrix}$. Does there exist a change of coordinates making this matrix diagonal? Why?

5. Assume that for a 2×2-matrix A we have $A^3 = 0$. Show that in that case we already have $A^2 = 0$. (*Hint*: one useful property you might want to use is $A^2 - \mathrm{tr}(A) \cdot A + \det(A) \cdot I_2 = 0$ proved in Homework 5.)