MA 1111: Linear Algebra I
Selected answers/solutions to the assignment for November 23, 2018

1. (**a**) Looking at the images of basis vectors, we immediately see that the matrix is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

(**b**) Using transition matrices, we immediately get $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}$.

2. This is a standard exercise on applying transition matrices. Answers are: (**a**) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$
(**b**) $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & -1 \\ -1 & 1 & 2 \end{pmatrix}$

3. (**a**) Induction: for $n = 0$ it is true; if $\begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix}^k \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b_k \\ b_{k+1} \end{pmatrix}$, then

$$\begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix}^{k+1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} b_k \\ b_{k+1} \end{pmatrix} = \begin{pmatrix} b_{k+1} \\ 3b_{k+1} - b_k \end{pmatrix} = \begin{pmatrix} b_{k+1} \\ b_{k+2} \end{pmatrix}.$$

(**b**) Eigenvalues are roots of $t^2 - 3t + 1 = 0$, i.e. $\lambda_1 = \frac{3+\sqrt{5}}{2}$, $\lambda_2 = \frac{3-\sqrt{5}}{2}$. The corresponding eigenvectors are $\begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$. Let $C = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}$. Then $C^{-1} \begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix} C = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, so $\begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix}^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} = C \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}^n C^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} \\ \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1 - \lambda_2} \end{pmatrix}$, and $b_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}$.

4. We have $\det(A - cI_3) = -c^3 + 4c^2 - 5c + 2 = -(c-1)^2(c-2)$, so the eigenvalues of this matrix are 1 and 2. Solving the systems of equations $Ax = x$ and $Ax = 2x$, we see that each solution is proportional to, respectively, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$. Thus, there is no basis of eigenvectors, and the answer to the question on change of coordinates is "no".

5. Note that $\det(A^3) = (\det(A))^3$, so if $A^3 = 0$, then $\det(A) = 0$. According to the previous question, we then have $A^2 - \mathrm{tr}(A)A = 0$, so $A^2 = \mathrm{tr}(A)A$. If $\mathrm{tr}(A) = 0$, we conclude that $A^2 = 0$. Otherwise, we have

$$0 = A^3 = A^2 \cdot A = \mathrm{tr}(A)A \cdot A = \mathrm{tr}(A)A^2 = (\mathrm{tr}(A))^2 \cdot A,$$

which for $\mathrm{tr}(A) \neq 0$ implies $A = 0$.