MA 1111: Linear Algebra I
Selected answers/solutions to the assignment for November 16, 2018

1. (**a**) True: if $\mathbf{v} = c_1\mathbf{v}_1 + \ldots + c_m\mathbf{v}_m$, then $(-1)\cdot\mathbf{v} + c_1\mathbf{v}_1 + \ldots + c_m\mathbf{v}_m = 0$, so the vectors $\mathbf{v}, \mathbf{v}_1, \ldots, \mathbf{v}_m$ are linearly dependent (already the first coefficient in the linear combination is nonzero).

(**b**) True: if a system of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ is linearly dependent, then $c_1\mathbf{v}_1 + \ldots + c_m\mathbf{v}_m = 0$ for some scalars $c_1, \ldots, c_m$, not all of which are zero. If $c_k \neq 0$, we have

$$v_k = \frac{-c_1}{c_k}\mathbf{v}_1 + \ldots + \frac{-c_{k-1}}{c_k}\mathbf{v}_{k-1} + \frac{-c_{k+1}}{c_k}\mathbf{v}_{k+1} + \ldots + \frac{-c_m}{c_k}\mathbf{v}_m.$$

(**c**) False: consider a system consisting of two vectors: $\mathbf{v} \neq 0$ and 0. Then $0\cdot\mathbf{v} + 1\cdot 0 = 0$, so these vectors are linearly dependent. However, $\mathbf{v} \neq c \cdot 0$, so not every vector in the system is equal to a linear combination of other vectors.

2. We have $c_1e_1 + c_2e_2 = \begin{pmatrix} 3c_1 + 5c_2 \\ 2c_1 + 3c_2 \end{pmatrix}$, so as we know from class, all the properties are related to properties of the matrix $A = \begin{pmatrix} 3 & 5 \\ 2 & 3 \end{pmatrix}$. The determinant of this matrix is nonzero, so the matrix A is invertible, and its reduced row echelon form is the identity matrix, so the vectors form a basis. Solving the system $Ax = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, we find the coordinates $x_1 = -8$, $x_2 = 5$.

3. $M_{ef} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 3 & 1 & 0 \\ -1 & -1 & 2 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -3/2 & -1 & 1/2 \\ 5/2 & 1 & -3/2 \\ 1/2 & 0 & 3/2 \end{pmatrix}$; coordinates are

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 7 \\ -3 \end{pmatrix} = \begin{pmatrix} 3/2 \\ -9/2 \\ 11/2 \end{pmatrix}$$

and

$$\begin{pmatrix} 3 & 1 & 0 \\ -1 & -1 & 2 \\ 1 & 0 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 7 \\ -3 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/2 \\ 7/2 \end{pmatrix}$$

respectively.

4. (**a**) We have $v \times (w_1 + w_2) = v \times w_1 + v \times w_2$ and $v \times (cw) = cv \times w$ by the known properties of cross products, so it is a linear operator. Also, $v\times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -2 \end{pmatrix}$, $v\times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $v \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$, so the matrix of our operator is $\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix}$.

(**b**) We have $v \cdot (w_1 + w_2) = v \cdot w_1 + v \cdot w_2$ and $v \cdot (cw) = cv \cdot w$ by the known properties of dot products, so it is a linear transformation. Also, $v \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1$, $v \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 2$, $v \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = -1$, so the matrix of our operator is $\begin{pmatrix} 1 & 2 & -1 \end{pmatrix}$.