MA 1111: Linear Algebra I

Selected answers/solutions to the assignment for November 2, 2018

**1.** (**a**) Inversions are only formed by $n$ and other numbers — $n-1$ in total, so the permutation is even for odd $n$ and odd for even $n$.

(**b**) *First solution*: this reordering of $1, \ldots, n$ is obtained from the normal ordering as a result of consecutive exchanges of $1$ with $n$, $2$ with $n-1$ etc. The total number of exchanges here is $\frac{n}{2}$ for an even $n$ and $\frac{n-1}{2}$ for an odd $n$. We know that exchanging two numbers changes the parity, so we need to determine the parity of $\frac{n}{2}$ for an even $n$ and of $\frac{n-1}{2}$ for an odd $n$. It is easy to see that the latter parity is determined by the remainder of $n$ modulo 4: for $n = 4k$ or $n = 4k+1$ the permutation is even, otherwise it is odd.

*Second solution*: every pair of numbers forms an inversion, so the total number of inversions is equal to $1+2+\ldots+(n-1) = n(n-1)/2$. To determine the parity of this number, we should know the remainder of $n$ modulo 4, and we easily obtain the answer given above.

**2.** (**a**) Subtracting from the row 3 of the matrix the rows 1 and 2, we get a matrix with a row of zeros, hence the determinant is equal to zero.

(**b**) We have $\det(AA^T) = \det(A)\det(A^T) = \det(A)^2$, so $\det(A) = 0$ if and only if $\det(AA^T) = 0$.

**3.** (**a**) If $P$ and $Q$ are $n \times n$-matrices, and $PQ - QP = I_n$, then

$$0 = \mathrm{tr}(PQ) - \mathrm{tr}(QP) = \mathrm{tr}(PQ - QP) = \mathrm{tr}(I_n) = n,$$

a contradiction.

(**b**) If $P$ is invertible, then $Q - P^{-1}QP = I_n$, and $0 = \mathrm{tr}(Q) - \mathrm{tr}(P^{-1}QP) = \mathrm{tr}(I_n) = n$, a contradiction.

**4.** (**a**) Yes, two vectors are linearly independent if they are not proportional, and these two clearly are not. (**b**) No, the sum of the second and the third is a multiple of the first one. (**c**) Yes (the matrix is invertible, so the RREF is $I_3$, and vectors are linearly independent). (**d**) No, because the maximal number of linearly independent vectors in $\mathbb{R}^3$ is 3 (proved in class).

**5.** (**a**) No, because the minimal number of vectors in a complete system in $\mathbb{R}^3$ is 3 (proved in class). (**b**) No: for each of these vectors the sum of coordinates is $0$, and the same will hold for every linear combination. (Alternative solution: compute the RREF and see that there is a row of zeros.) (**c**) Yes (the matrix is invertible, so the RREF is $I_3$, and vectors form a complete system). (**d**) Yes (easy to check that the RREF of the matrix has no zero rows so the system is complete).