MA 1111: Linear Algebra I

Selected answers/solutions to the assignment for October 19, 2018

1. (**a**) $M_{11}=12$, $M_{12}=-4$, $M_{13}=-8$, $M_{21}=19$, $M_{22}=7$, $M_{23}=-10$, $M_{31}=9$, $M_{32}=5$, $M_{33}=2$.

(**b**) $C_{11}=12$, $C_{12}=4$, $C_{13}=-8$, $C_{21}=-19$, $C_{22}=7$, $C_{23}=10$, $C_{31}=9$, $C_{32}=-5$, $C_{33}=2$.

(**c**) $2\cdot 12+4\cdot 4-8=32$; (**d**) $2\cdot 12-19+3\cdot 9=32$; (**e**) $-19+3\cdot 7+3\cdot 10=32$; (**f**) $4\cdot 4+3\cdot 7-5=32$; (**g**) $3\cdot 9-5+5\cdot 2=32$; (**h**) $1\cdot(-8)+3\cdot 10+5\cdot 2=32$.

2. (**a**) The adjugate matrix is equal to $\begin{pmatrix}12 & -19 & 9\\ 4 & 7 & -5\\ -8 & 10 & 2\end{pmatrix}$, and the inverse matrix is $\frac{1}{32}\begin{pmatrix}12 & -19 & 9\\ 4 & 7 & -5\\ -8 & 10 & 2\end{pmatrix}$. (**b**) $\det\begin{pmatrix}1 & 4 & 1\\ -1 & 3 & 3\\ 1 & 1 & 5\end{pmatrix}=40$ $\det\begin{pmatrix}2 & 1 & 1\\ 1 & -1 & 3\\ 3 & 1 & 5\end{pmatrix}=-8$ $\det\begin{pmatrix}2 & 4 & 1\\ 1 & 3 & -1\\ 3 & 1 & 1\end{pmatrix}=-16$, so by Cramer's rule the only solution to this system is $\begin{pmatrix}5/4\\ -1/4\\ -1/2\end{pmatrix}$.

3. (**a**) If B is not invertible, then the system $Bx=0$ has a nontrivial solution, so there exists a column $y\neq 0$ such that $By=0$. We have $ABy=A0=0$, so the system $ABx=0$ has a nontrivial solution, and AB is not invertible, a contradiction. (**b**) We have $A=ABB^{-1}$, so A is invertible as a product of invertible matrices.

4. (**a**) We have $\mathrm{tr}(M^{-1}NM)=\mathrm{tr}((M^{-1}N)M)=\mathrm{tr}(M(M^{-1}N))=\mathrm{tr}(MM^{-1}N)=\mathrm{tr}(N)$.

(**b**) We have

$$\det(M^{-1}NM)=\det(M^{-1})\det(N)\det(M)=\det(M^{-1})\det(M)\det(N)=\det(M^{-1}MN)=\det(N).$$

5. Let $A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$; we have $A^2=\begin{pmatrix}a^2+bc & b(a+d)\\ c(a+d) & d^2+bc\end{pmatrix}$, so

$$A^2-\mathrm{tr}(A)A+\det(A)I_2=$$
$$=\begin{pmatrix}a^2+bc & b(a+d)\\ c(a+d) & d^2+bc\end{pmatrix}-\begin{pmatrix}(a+d)a & (a+d)b\\ (a+d)c & (a+d)d\end{pmatrix}+\begin{pmatrix}ad-bc & 0\\ 0 & ad-bc\end{pmatrix}=0.$$