MA 1111: Linear Algebra I
Selected answers/solutions to the assignment for October 12, 2018

1. (**a**) Even (12 inversions); (**b**) odd (17 inversions); (**c**) odd (17 inversions).

2. We should have $k = 4$ (to have all integers from 1 to 6 in the top row; also, the numbers i, j, l should be equal (in some order) to $2, 4, 5$. If $i = 2$, $j = 4$, $l = 5$, we get the permutation $\begin{pmatrix} 5 & 2 & 4 & 3 & 6 & 1 \\ 5 & 1 & 3 & 2 & 6 & 4 \end{pmatrix}$ which is odd. Other permutations which occur correspond to $i = 4$, $j = 2$, $l = 5$ (even, because we exchange one pair of numbers in an odd permutation), $i = 2$, $j = 5$, $l = 4$ (even, because we exchange one pair of numbers in an odd permutation), $i = 5$, $j = 4$, $l = 2$ (even, because we exchange one pair of numbers in an odd permutation), $i = 4$, $j = 5$, $l = 2$ (odd, because we exchange two pairs of numbers in an odd permutation), $i = 5$, $j = 2$, $l = 4$ (odd, because we exchange two pairs of numbers in an odd permutation). Overall, the answer is $(i, j, k, l) = (4, 2, 4, 5)$, $(i, j, k, l) = (2, 5, 4, 4)$, or $(i, j, k, l) = (5, 4, 4, 2)$.

3. (**a**) Performing elementary row operations, we get

$$\det\begin{pmatrix} 1 & 0 & -2 \\ 1 & 1 & 3 \\ 4 & 3 & 1 \end{pmatrix} = \det\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 5 \\ 0 & 3 & 9 \end{pmatrix} = \det\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 5 \\ 0 & 0 & -6 \end{pmatrix} = -6$$

(**b**) Performing elementary row operations, we get

$$\det\begin{pmatrix} 1 & 1 & -2 & -1 \\ 2 & 0 & 3 & -1 \\ 4 & 2 & 3 & 1 \\ 3 & 0 & 0 & 1 \end{pmatrix} = \det\begin{pmatrix} 1 & 1 & -2 & -1 \\ 0 & -2 & 7 & 1 \\ 0 & -2 & 11 & 5 \\ 0 & -3 & 6 & 4 \end{pmatrix} = \det\begin{pmatrix} 1 & 1 & -2 & -1 \\ 0 & -2 & 7 & 1 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & -9/2 & 5/2 \end{pmatrix} =$$

$$= 4/2\det\begin{pmatrix} 1 & 1 & -2 & -1 \\ 0 & -2 & 7 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -9 & 5 \end{pmatrix} = 2\det\begin{pmatrix} 1 & 1 & -2 & -1 \\ 0 & -2 & 7 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 14 \end{pmatrix} = -56.$$

4. (**a**) Performing elementary row operations, we get

$$\det\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} = \det\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix} = \det\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = 1.$$

(**b**) Performing elementary row operations, we get

$$\det\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix} = \det\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \det\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \det\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 1.$$

(**c**) Subtracting from each of the rows the previous row (from the row n up), we get the matrix

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & 1 & 1 & \dots & 1 & 1 \\ 0 & 0 & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & 1 & 1 \\ 0 & 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix},$$

so the determinant of our matrix is equal to 1.

5. (**a**) $\det(A) = (2-c)^2 - 1 = c^2 - 4c + 3 = (c-1)(c-3)$, so the matrix is not invertible for $c = 1$ and $c = 3$.

(**b**) Performing elementary row operations, we get

$$\det(A) = -\det \begin{pmatrix} 1 & c-1 & 2 \\ 3 & 2 & 1+c \\ -1 & 4c & 3 \end{pmatrix} = -\det \begin{pmatrix} 1 & c-1 & 2 \\ 0 & 5-3c & c-5 \\ 0 & 5c-1 & 5 \end{pmatrix} =$$

$$= -5\det \begin{pmatrix} 1 & c-1 & 2 \\ 0 & 5-3c & c-5 \\ 0 & c-1/5 & 1 \end{pmatrix} = -5\det \begin{pmatrix} 1 & c-1 & 2 \\ 0 & 5-3c-(c-1/5)(c-5) & 0 \\ 0 & c-1/5 & 1 \end{pmatrix} =$$

$$= -5(5-3c-(c-1/5)(c-5)) \det \begin{pmatrix} 1 & c-1 & 2 \\ 0 & 1 & 0 \\ 0 & c-1/5 & 1 \end{pmatrix} =$$

$$= (5c^2 - 11c - 20) \det \begin{pmatrix} 1 & c-1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (5c^2 - 11c - 20),$$

so A is not invertible when c is a root of $5c^2 - 11c - 20 = 0$, that is $\frac{11 \pm \sqrt{521}}{10}$.