MA 1111: Linear Algebra I
Selected answers/solutions to the assignment for October 5, 2018

1. (**a**) $A+B$ and BA are not defined, $AB=\begin{pmatrix}2\\1\end{pmatrix}$;

(**b**) $A+B$ and BA are not defined, $AB=\begin{pmatrix}7\\4\end{pmatrix}$;

(**c**) $A+B$ is not defined, $BA=\begin{pmatrix}9&5&8\\1&1&2\\15&7&10\end{pmatrix}$, $AB=\begin{pmatrix}5&21\\5&15\end{pmatrix}$;

(**d**) $A+B=\begin{pmatrix}4&8\\3&3\end{pmatrix}$, $BA=\begin{pmatrix}8&12\\6&10\end{pmatrix}$, $AB=\begin{pmatrix}12&16\\4&6\end{pmatrix}$.

2. The easiest thing to do is to apply the algorithm from the lecture: take the matrix $(A \mid I_n)$ and bring it to the reduced row echelon form; the result is $(I_n \mid A^{-1})$ if the matrix is invertible, and has $(R \mid B)$ with $R \neq I_n$ otherwise.

(**a**) $\begin{pmatrix}5&3\\2&1\end{pmatrix}$ is invertible, the inverse is $\begin{pmatrix}-1&3\\2&-5\end{pmatrix}$;

(**b**) $\begin{pmatrix}6&3\\2&1\end{pmatrix}$ is not invertible, since the reduced row echelon form of $(A \mid I)$ is $\begin{pmatrix}1&1/2&0&1/2\\0&0&1&-3\end{pmatrix}$, and the matrix on the left is not the identity;

(**c**) $\begin{pmatrix}2&1&0\\1&2&0\end{pmatrix}$ is not invertible, since in class we proved that only square matrices are invertible;

(**d**) $\begin{pmatrix}1&2&3\\2&3&4\\1&3&9\end{pmatrix}$ is invertible; the inverse is $\begin{pmatrix}-15/4&9/4&1/4\\7/2&-3/2&-1/2\\-3/4&1/4&1/4\end{pmatrix}$.

3. (**a**) Suppose that A is a $k\times l$-matrix, and B is an $m\times n$-matrix. In order for AB to be defined, we must have $l=m$. In order for BA to be defined, we must have $n=k$. Consequently, the size of matrix AB is $k\times n=n\times n$, and the size of the matrix BA is $m\times l=m\times m$, which is exactly what we want to prove.

(**b**) We have

$$\operatorname{tr}(UV)=(UV)_{11}+(UV)_{22}+\ldots+(UV)_{nn}=(U_{11}V_{11}+U_{12}V_{21}+\ldots+U_{1n}V_{n1})+\\(U_{21}V_{12}+U_{22}V_{22}+\ldots+U_{2n}V_{n2})+\ldots+(U_{n1}V_{1n}+U_{n2}V_{2n}+\ldots+U_{nn}V_{nn}),$$

and

$$\operatorname{tr}(VU)=(VU)_{11}+(VU)_{22}+\ldots+(VU)_{nn}=(V_{11}U_{11}+V_{12}U_{21}+\ldots+V_{1n}U_{n1})+\\(V_{21}U_{12}+V_{22}U_{22}+\ldots+V_{2n}U_{n2})+\ldots+(V_{n1}U_{1n}+V_{n2}U_{2n}+\ldots+V_{nn}U_{nn}),$$

so both traces are actually equal to the sum of all products $U_{ij}V_{ji}$, where i and j range from 1 to n. For the example $U=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $V=\begin{pmatrix}0&0\\1&0\end{pmatrix}$ from class, we have $UV=\begin{pmatrix}1&0\\0&0\end{pmatrix}$ and $VU=\begin{pmatrix}0&0\\0&1\end{pmatrix}$, so even though these matrices are not equal, their traces are equal (they both are equal to 1).

4. (**a**) For example, $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ would work.

(**b**) The reduced row echelon form R of A must have a row of zeros, and therefore the product RB will have a row of zeros, and cannot be invertible, but RB is obtained from AB by elementary row operations, so it must be invertible if $AB = I_3$ is invertible.