MA 1111: Linear Algebra I

Selected answers/solutions to the assignment for September 28, 2018

1. (**a**) Denoting by $\mathbf{r} = (x, y, z)$ the vector from the origin to a generic point (x, y, z) in this plane, and by $\mathbf{u}$ the vector from the origin to $(1, -1, 1)$, we get $(\mathbf{r} - \mathbf{u}) \cdot \mathbf{n} = 0$, so $(x-1) - 2(y+1) + (z-1) = 0$, or $x - 2y + z = 4$.

(**b**) Denoting the vectors from the origin to these points by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, we note that for a perpendicular vector, we can take $(\mathbf{v}-\mathbf{u}) \times (\mathbf{w}-\mathbf{u}) = (6, 2, 7)$. Now, analogously to the previous question, we obtain

$$6(x-1) + 2(y+1) + 7(z-1) = 0,$$

or $6x + 2y + 7z = 11$.

2. We have $x = \frac{1+y-4z}{2}$, so the second equation becomes $\frac{7}{2} + \frac{7}{2}y - 14z + 2y + z = 5$, $\frac{11}{2}y - 13z = \frac{3}{2}$, $y = \frac{2}{11}(\frac{3}{2} + 13z)$. If we let z equal to an arbitrary value t, we get $x = \frac{1 + \frac{3}{11} + \frac{26}{11}t - 4t}{2} = \frac{7}{11} - \frac{9}{11}t$. Thus,

$$\begin{aligned} x &= \frac{7}{11} - \frac{9}{11}t, \\ y &= \frac{3}{11} + \frac{26}{11}t, \\ z &= t. \end{aligned}$$

3. The system of equations is

$$\begin{cases} x_1 + 4x_2 + 5x_3 + x_4 = 1, \\ x_1 + 2x_2 + 2x_3 + x_4 = -4, \\ x_1 + 2x_2 + 5x_4 = -4. \end{cases}$$

Below, the solution is phrased using transformations of equations; alternatively, one can perform same operations on matrices. We have

$$\begin{cases} x_1 + 4x_2 + 5x_3 + x_4 = 1, \\ x_1 + 2x_2 + 2x_3 + x_4 = -4, \\ x_1 + 2x_2 + 5x_4 = -4. \end{cases} \overset{(2)-(1),(3)-(1)}{\longmapsto} \begin{cases} x_1 + 4x_2 + 5x_3 + x_4 = 1, \\ -2x_2 - 3x_3 = -5, \\ -2x_2 - 5x_3 + 4x_4 = -5, \end{cases} \overset{-1/2\times(2)}{\longmapsto}$$

$$\begin{cases} x_1 + 4x_2 + 5x_3 + x_4 = 1, \\ x_2 + \frac{3}{2}x_3 = \frac{5}{2}, \\ -2x_2 - 5x_3 + 4x_4 = -5, \end{cases} \overset{(1)-4(2),(3)+2(2)}{\longmapsto} \begin{cases} x_1 - x_3 + x_4 = -9, \\ x_2 + \frac{3}{2}x_3 = \frac{5}{2}, \\ -2x_3 + 4x_4 = 0, \end{cases} \overset{-1/2\times(3)}{\longmapsto}$$

$$\begin{cases} x_1 - x_3 + x_4 = -9, \\ x_2 + \frac{3}{2}x_3 = \frac{5}{2}, \\ x_3 - 2x_4 = 0, \end{cases} \overset{(1)+(3),(2)-3/2\times(3)}{\longmapsto} \begin{cases} x_1 - x_4 = -9, \\ x_2 + 3x_3 = \frac{5}{2}, \\ x_3 - 2x_4 = 0, \end{cases}$$

We see that x_4 can be assigned an arbitrary value t, and the general solution to this system is

$$\begin{aligned} x_1 &= -9 + t \\ x_2 &= 5/2 - 3t \\ x_3 &= 2t \\ x_4 &= t \end{aligned}$$

where t is any number.

4. We have $x = 1 - b - 4ay$, and substituting that into the second equation we get

$$a(1 - b - 4ay) + y = b,$$

so $y(1 - 4a^2) = b - a + ab$. This means that for $a \neq \pm\frac{1}{2}$ and every b the system has exactly one solution, for $a = \frac{1}{2}$ we have a condition $b - \frac{1}{2} + \frac{1}{2}b = 0$ for the system to have solutions (this reads $b = \frac{1}{3}$, so for $a = \frac{1}{2}$, $b = \frac{1}{3}$ there are infinitely many solutions, and for $a = \frac{1}{2}$, $b \neq \frac{1}{3}$ the system has no solutions), and for $a = -\frac{1}{2}$ we have the condition $b + \frac{1}{2} - \frac{1}{2}b = 0$ for the system to have solutions (this reads $b = -1$, so for $a = -\frac{1}{2}$, $b = -1$ there are infinitely many solutions, and for $a = -\frac{1}{2}$, $b \neq -1$ the system has no solutions).