1111: Linear Algebra I

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Towards computing determinants

In school, you may have seen either the formula

$$A^{-1} = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

for a 2×2-matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, or at least its consequence, the formula

$$\begin{cases} x = \frac{de-bf}{ad-bc}, \\ y = \frac{af-ce}{ad-bc}, \end{cases}$$

allowing to solve a system of two equations with two unknowns

$$\begin{cases} ax + by = e, \\ cx + dy = f. \end{cases}$$

Now, we shall see how to generalise these formulas for $n \times n$-matrices.

Permutations

To proceed, we need to introduce the notion of a *permutation*. By definition, a permutation of n elements is a rearrangement of numbers $1, 2, \ldots, n$ in a particular order.

For example, $1, 3, 4, 2$ is a permutation of four elements, and $1, 4, 3, 4, 2$ is not (because the number 4 is repeated).

We shall also use the two-row notation for permutations: a permutation of n elements may be represented by a $2 \times n$-matrix made up of columns $\begin{pmatrix} j \\ a_j \end{pmatrix}$, where a_j is the number at the j-th place in the permutation.

For example, the permutation $1, 3, 4, 2$ may be represented by the matrix $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$, but also by the matrix $\begin{pmatrix} 1 & 4 & 3 & 2 \\ 1 & 2 & 4 & 3 \end{pmatrix}$, and by many other matrices.

Incidentally, the number of different permutations of n elements is equal to $1 \cdot 2 \cdots n$, this number is called "n factorial" and is denoted by $n!$ (n with an exclamation mark).

Odd and even permutations

Let σ be a permutation of n elements, written in the one-row notation. Two numbers i and j, where $1 \le i < j \le n$, are said to form an inversion in σ, if they are "listed in wrong order", that is j appears before i in σ.

For the permutation $1, 3, 4, 2$, there are 6 pairs (i, j) to look at: $(1, 2)$, $(1, 3)$, $(1, 4)$, $(2, 3)$, $(2, 4)$, $(3, 4)$. Of these, the pair $(2, 3)$ forms an inversion, and the pair $(2, 4)$ does, and other pairs do not.

A permutation is said to be *even* if its number of inversions is even, and *odd* otherwise. One of the most important properties of this division into even and odd is the following: *if we swap two numbers in a permutation $a_1, \ldots, a_n$, it makes an even permutation into an odd one, and vice versa*.

Let us first remark that this is obvious if we swap two neighbours, a_p and a_{p+1}. Indeed, this only changes whether they form an inversion or not, since their positions relative to others do not change. Now, a swap of a_p and a_q can be done by dragging a_p through $a_{p+1}, \ldots, a_{q-1}$, swapping it with a_q, and dragging a_q through $a_{q-1}, \ldots, a_{p+1}$, so altogether we do an odd number of "swapping neighbours".

Odd and even permutations

The property that we just proved is useful for a yet another definition of even / odd permutations that refers to the two-row notation. Namely, a permutation in two-row notation is even if the total number of inversions in the top and the bottom row is even, and is odd, if the total number of inversions in the top and the bottom row is odd.

The usual problem to address is whether this definition makes sense or not: maybe it will give different answers for different two-row representations. Luckily, it is not the case: different representations are obtained from one another by re-arranging columns, and each swap of columns will change the number of inversions in the top row and in the bottom row by odd numbers, so altogether will give a change by an even number.

For example, the two different representations of $1, 3, 4, 2$, that we discussed, everything works: for $\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 3 & 4 & 2\end{pmatrix}$, the total number of inversions is 2 and for $\begin{pmatrix}1 & 4 & 3 & 2\\ 1 & 2 & 4 & 3\end{pmatrix}$ the total number of inversions is $3 + 1 = 4$.

Determinants

We shall now define an important numeric invariant of an $n \times n$-matrix A, the *determinant* of A, denoted $\det(A)$. Informally, the determinant of A is the signed sum of all possible products of n entries of A, chosen in a way that every row and every column is represented in the product exactly once.

Formally,

$$\det(A) = \sum_{\sigma} \mathrm{sign}(\sigma) A_{i_1 j_1} A_{i_2 j_2} \cdots A_{i_n j_n} \ .$$

Here σ runs over all permutations of n elements, and $\begin{pmatrix} i_1 & i_2 & \ldots & i_n \\ j_1 & j_2 & \ldots & j_n \end{pmatrix}$ is some two-row representation of σ. The *sign* $\mathrm{sign}(\sigma)$ of a permutation σ is defined to be 1 if σ is even, and -1 if σ is odd.

Properties of determinants

Let me list some properties of determinants that we shall eventually prove. Even before we prove them, you can use them in calculations without proof.

- If three matrices A, A', and A'' have all rows except for the i-th row i in common, and the i-th row of A is equal to the sum of the i-th rows of A' and A'', then $\det(A) = \det(A') + \det(A'')$;
- if two matrices A and A' have all rows except for the i-th row in common, and the i-th row of A' is obtained from the i-th row of A by multiplying it by a scalar c, then $\det(A') = c\det(A)$;
- if two matrices A and A' have all rows except for the i-th and the j-th row in common, and A' is obtained from the A by swapping the i-th row with the j-th row, then $\det(A') = -\det(A)$;
- if two matrices A and A' have all rows except for the i-th row in common, and the i-th row of A' is obtained from the i-th row of A by adding a multiple of another row, then $\det(A') = \det(A)$;
- for each n, we have $\det(I_n) = 1$.

Properties of determinants

Effectively, these properties say that

- determinants are *multilinear functions* of their rows,
- determinants behave predictably with respect to elementary row operations.

Let us give an example of how this can be used:

$$\det\begin{pmatrix}1 & 2 & 2\\ 1 & 1 & 4\\ 1 & 2 & 8\end{pmatrix} \overset{(2)-(1),(3)-(1)}{=} \det\begin{pmatrix}1 & 2 & 2\\ 0 & -1 & 2\\ 0 & 0 & 6\end{pmatrix} =$$

$$(-1)\cdot 6\cdot\det\begin{pmatrix}1 & 2 & 2\\ 0 & 1 & -2\\ 0 & 0 & 1\end{pmatrix} \overset{(2)+2(3),(1)-2(3)}{=} -6\det\begin{pmatrix}1 & 2 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \overset{(1)-2(2)}{=}$$

$$-6\det\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = -6.$$