1111: Linear Algebra I

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Lecture 22

Computing Fibonacci numbers

As we discussed previously, a matrix of a linear operator φ is diagonal in the system of coordinates given by the basis e_1, e_2 if and only if $\varphi(e_1) = a_1 e_1$, $\varphi(e_2) = a_2 e_2$, or, in general in dimension n, if $\varphi(e_1) = a_1 e_1$, ..., $\varphi(e_n) = a_n e_n$.

Suppose that v is a nonzero vector for which $\varphi(v) = av$ for some scalar a. For whichever basis $e_1, \ldots, e_n$ we may have, this means $A_{\varphi,\mathbf{e}} v_{\mathbf{e}} = a v_{\mathbf{e}}$, or

$$(A_{\varphi,v_{\mathbf{e}}} - aI_n)v_{\mathbf{e}} = 0.$$

This means that the matrix $A = A_{\varphi,v_{\mathbf{e}}} - aI_n$ is not invertible, and that $\det(A) = 0$. Note that $\det(A)$ is a polynomial expression in A of degree n. For example, for $A_{\varphi,\mathbf{e}} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, we have

$$\det(A) = a^2 - a(a_{11} + a_{22}) + a_{11}a_{22} - a_{12}a_{21} = a^2 - \operatorname{tr}(A_{\varphi,\mathbf{e}}) + \det(A_{\varphi,\mathbf{e}}),$$

the polynomial equation we obtained before in a different way.

Therefore, the vectors that are reasonable candidates for a basis are obtained from solutions of the systems of equations $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} x = \frac{1\pm\sqrt{5}}{2}x$. The first of them has the general solution $\begin{pmatrix} x_1 \\ \frac{1+\sqrt{5}}{2}x_1 \end{pmatrix}$, and the second one has the general solution $\begin{pmatrix} x_1 \\ \frac{1-\sqrt{5}}{2}x_1 \end{pmatrix}$. Setting in each cases $x_1 = 1$, we obtain two vectors $\mathbf{e}_1 = \begin{pmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{pmatrix}$ and $\mathbf{e}_2 = \begin{pmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{pmatrix}$. The transition matrix from the basis of standard unit vectors $\mathbf{s}_1$, $\mathbf{s}_2$ to this basis is, manifestly, $M_{\mathbf{s},\mathbf{e}} = \begin{pmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{pmatrix}$, so

$$M_{\mathbf{s},\mathbf{e}}^{-1} = \left(-\frac{1}{\sqrt{5}}\right) \begin{pmatrix} \frac{1-\sqrt{5}}{2} & -1 \\ \frac{-1-\sqrt{5}}{2} & 1 \end{pmatrix},$$

Since $A\mathbf{e}_1 = (\frac{1+\sqrt{5}}{2})\mathbf{e}_1$, and $A\mathbf{e}_2 = (\frac{1-\sqrt{5}}{2})\mathbf{e}_2$, the matrix of the linear transformation φ relative to the basis $\mathbf{e}_1$, $\mathbf{e}_2$ is

$$M_{\mathbf{s},\mathbf{e}}^{-1} A M_{\mathbf{s},\mathbf{e}} = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix}.$$

Therefore,

$$A = M_{\mathbf{s},\mathbf{e}} \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix} M_{\mathbf{s},\mathbf{e}}^{-1},$$

and hence

$$A^n = \left(M_{\mathbf{s},\mathbf{e}} \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix} M_{\mathbf{s},\mathbf{e}}^{-1} \right)^n = M_{\mathbf{s},\mathbf{e}} \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix}^n M_{\mathbf{s},\mathbf{e}}^{-1} = M_{\mathbf{s},\mathbf{e}} \begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{pmatrix} M_{\mathbf{s},\mathbf{e}}^{-1}.$$

Substituting the above formulas for $M_{\mathbf{s},\mathbf{e}}$ and $M_{\mathbf{s},\mathbf{e}}^{-1}$, we see that

$$A^n = \begin{pmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{pmatrix} \left(-\frac{1}{\sqrt{5}}\right) \begin{pmatrix} \frac{1-\sqrt{5}}{2} & -1 \\ \frac{-1-\sqrt{5}}{2} & 1 \end{pmatrix}$$

In fact, we have $\mathbf{v}_n = A^n \mathbf{v}_0$, so

$$\begin{aligned} \mathbf{v}_n &= \begin{pmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{pmatrix} \left(-\frac{1}{\sqrt{5}}\right) \begin{pmatrix} \frac{1-\sqrt{5}}{2} & -1 \\ \frac{-1-\sqrt{5}}{2} & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \\ &= \begin{pmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n \\ -\frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n \end{pmatrix} = \\ &= \begin{pmatrix} \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \right) \\ \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \right) \end{pmatrix} \end{aligned}$$

Recalling that $\mathbf{v}_n = \begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix}$, we observe that

$$f_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \right).$$

This formula is quite informative. For instance, we can remark that $\left|\frac{1-\sqrt{5}}{2}\right| < 1$, so for large n the Fibonacci number f_n is the closest integer to $\frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n$.

Next week, we shall discuss some further examples of applications of linear algebra.