1111: Linear Algebra I

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Lecture 19

Linear maps and transformations

Example 1. Let V be the vector space of all polynomials in one variable x. Consider the function $\alpha\colon V \to V$ that maps every polynomial $f(x)$ to $3f(x)f'(x)$. This is not a linear map; for example, $1 \mapsto 0$, $x \mapsto 3x$, but $x + 1 \mapsto 3(x+1) = 3x + 3 \neq 3x + 0$.

Example 2. Consider the vector space M_2 of all 2×2-matrices. Let us define a function $\beta\colon M_2 \to M_2$ by the formula $\beta(X) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} X$. Let us check that this map is a linear transformation. Indeed, by properties of matrix products

$$\beta(X_1 + X_2) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}(X_1 + X_2) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} X_1 + \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} X_2 = \beta(X_1) + \beta(X_2),$$
$$\beta(cX) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}(cX) = c\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} X = c\beta(X).$$

Lemma 1. *Suppose that f is a linear map. Then $f(0) = 0$, and $f(-v) = -f(v)$.*

Proof. This follows from $0 \cdot v = 0$ and $(-1) \cdot v = -v$. □

Definition 1. Let $\varphi\colon V \to W$ be a linear map, and let $e_1, \ldots, e_n$ and $f_1, \ldots, f_m$ be bases of V and W respectively. Let us compute coordinates of the vectors $\varphi(e_i)$ with respect to the basis $f_1, \ldots, f_m$:

$$\varphi(e_1) = a_{11}f_1 + a_{21}f_2 + \cdots + a_{m1}f_m,$$
$$\varphi(e_2) = a_{12}f_1 + a_{22}f_2 + \cdots + a_{m2}f_m,$$
$$\ldots$$
$$\varphi(e_n) = a_{1n}f_1 + a_{2n}f_2 + \cdots + a_{mn}f_m.$$

The matrix

$$A_{\varphi,\mathbf{e},\mathbf{f}} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \ldots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$$

is called *the matrix of the linear map φ with respect to the given bases.* For each k, its k-th column is the column of coordinates of image $\varphi(e_k)$.

Similarly to how we proved it for transition matrices, we have the following result.

Lemma 2. *Let $\varphi\colon V \to W$ be a linear operator, and let $e_1, \ldots, e_n$ and $f_1, \ldots, f_m$ be bases of V and W respectively. Suppose that $x_1, \ldots, x_n$ are coordinates of some vector v relative to the basis $e_1, \ldots, e_n$, and*

$y_1, \ldots, y_m$ *are coordinates of* $\varphi(v)$ *relative to the basis* $f_1, \ldots, f_m$. *In the notation above, we have*

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = A_{\varphi,\mathbf{e},\mathbf{f}} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Proof. The proof is indeed very analogous to the one for transition matrices: we have

$$v = x_1 e_1 + \cdots + x_n e_n,$$

so that

$$\varphi(v) = x_1 \varphi(e_1) + \cdots + x_n \varphi(e_n).$$

Substituting the expansion of $f(e_i)$'s in terms of f_j's, we get

$$\begin{aligned} \varphi(v) = x_1(a_{11}f_1 + a_{21}f_2 + \cdots + a_{m1}f_m) + \cdots + x_n(a_{1n}f_1 + a_{2n}f_2 + \cdots + a_{mn}f_m) = \\ = (a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n)f_1 + \cdots + (a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n)f_n. \end{aligned}$$

Since we know that coordinates are uniquely defined, we conclude that

$$\begin{gathered} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = y_1, \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = y_n, \end{gathered}$$

which is what we want to prove. □

Example 3. Let us consider the linear map $X\colon P_2 \to P_3$ discussed in previous class. Let us take the bases $e_1 = 1, e_2 = x, e_3 = x^2$ of P_2, and the basis $f_1 = 1, f_2 = x, f_3 = x^2, f_4 = x^3$ of P_3, and compute $A_{X,\mathbf{e},\mathbf{f}}$. Note that $X(e_1) = x \cdot 1 = x = f_2$, $X(e_2) = x \cdot x = x^2 = f_3$, and $X(e_3) = x \cdot x^2 = x^3 = f_4$. Therefore

$$A_{X,\mathbf{e},\mathbf{f}} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Example 4. Let us consider the linear map $D\colon P_3 \to P_3$ and $\hat{D}\colon P_3 \to P_3$ discussed in the previous class. Let us take the bases $e_1 = 1, e_2 = x, e_3 = x^2, e_4 = x^3$ of P_3, and the basis $f_1 = 1, f_2 = x, f_3 = x^2$ of P_2, and let us compute $A_{D,\mathbf{e},\mathbf{f}}$ and $A_{\hat{D},\mathbf{e}}$. Note that $D(e_1) = 1' = 0$, $D(e_2) = x' = 1 = f_1$, $D(e_3) = (x^2)' = 2x = 2f_2$, and $D(e_4) = (x^3)' = 3x^2 = 3f_3$, and that $\hat{D}(e_1) = 1' = 0$, $\hat{D}(e_2) = x' = 1 = e_1$, $\hat{D}(e_3) = (x^2)' = 2x = 2e_2$, and $\hat{D}(e_4) = (x^3)' = 3x^2 = 3e_3$. Therefore

$$A_{D,\mathbf{e},\mathbf{f}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

and

$$A_{\hat{D},\mathbf{e},\mathbf{e}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Example 5. Let us look at the linear map $\alpha\colon M_2 \to M_2$ discussed in the beginning of this class. We consider the basis of matrix units in M_2: $e_1 = E_{11}$, $e_2 = E_{12}$, $e_3 = E_{21}$, $e_4 = E_{22}$. We have

$$\begin{aligned}
\alpha(e_1) &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = e_1 + e_3, \\
\alpha(e_2) &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} e_2 + e_4, \\
\alpha(e_3) &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = e_1, \\
\alpha(e_4) &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = e_2,
\end{aligned}$$

so

$$A_{\alpha,\mathbf{e}} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

The next statement is also similar to the corresponding one for transition matrices; it also generalises the statement that in the case of coordinate vector spaces product of matrices corresponds to composition of linear maps.

Lemma 3. *Let U, V, and W be vector spaces, and let $\psi\colon U \to V$ and $\varphi\colon V \to W$ be linear operators. Finally, let $e_1, \ldots, e_n$, $f_1, \ldots, f_m$, and $g_1, \ldots, g_k$ be bases of U, V, and W respectively. Then*

$$A_{\varphi\circ\psi,\mathbf{e},\mathbf{g}} = A_{\varphi,\mathbf{f},\mathbf{g}} A_{\psi,\mathbf{e},\mathbf{f}}.$$

We shall prove it in the next class.