1111: Linear Algebra I

Dr. Vladimir Dotsenko (Vlad)

Lecture 15

More examples. Subspaces

Example 1. The set of all polynomials

$$a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0$$

in one variable x with real coefficients is a vector space with respect to addition and re-scaling. If we consider polynomials of degree at most n for some given n, this is also a vector space. Polynomials of degree exactly n do not form a vector space, since the sum of two such polynomials may have smaller degree, e.g. $x^n + (1 - x^n) = 1$, where the sum of two polynomials of degree n is of degree 0.

Definition 1. A subset U of a vector space V is called a subspace if it contains 0, is closed under addition and scalar multiplication, and is closed under taking negatives.

As in the case of $\mathbb{R}^n$, a subspace of a vector space is again a vector space.

Example 2. The set of all polynomials is a subspace of $C([0, 1])$, since polynomials are continuous functions.

Consequences of properties of vector operations

The properties 1-8 altogether allow to operate with elements of V as though they were vectors in $\mathbb{R}^n$, that is create linear combinations, take summands in an equation from the left hand side to the right hand side with opposite signs, collect similar terms etc. For that reason, we shall refer to elements of an abstract vector space as vectors, and to real numbers as scalars.

These properties also allow to prove various theoretical statements about vectors. There will be some of those in your next homework, and for now let me give several examples.

Lemma 1. *For all $v \in V$, we have $0 \cdot v = 0$.*

Proof. Denote $u = 0 \cdot v$. We have $u + u = 0 \cdot v + 0 \cdot v = (0 + 0 \cdot v) = 0 \cdot v = u$ (we used property 6 in the middle equality). But now we can "subtract u from both sides": $(u + u) + (-u) = u + (-u) = 0$ (property 4). Finally, $(u + u) + (-u) = u + (u + (-u)) = u + 0 = u$ (properties 1, 4, and 3). We conclude that $u = 0$. □

The following lemma proved similarly with property 5 instead of property 6:

Lemma 2. *For all $c \in \mathbb{R}$, we have $c \cdot 0 = 0$.*

Let us prove another statement that is sometimes useful.

Lemma 3. *Suppose that for a scalar c and a vector v we have $c \cdot v = 0$. Then $c = 0$ or $v = 0$.*

Proof. If $c = 0$ there is nothing to prove. Suppose $c \neq 0$. Then $0 = c^{-1} \cdot 0 = c^{-1}(c \cdot v) = (c^{-1}c)v = 1 \cdot v = v$ (by Lemma 2 above, and properties 7 and 8). Therefore, $v = 0$, as required. □

There are some further properties that you will get as exercises in forthcoming homeworks.

Fields

It is also worth mentioning that sometimes we shall use other scalars, not just real numbers. In order for all the arguments to work, we need that scalars have arithmetics similar to that of real numbers. Let us be precise about what that means.

Definition 2. A *field* is a set F equipped with the following data:

- a rule assigning to each elements $f_1, f_2 \in F$ an element of F denoted $v_1 + v_2$, and
- a rule assigning to each elements $f_1, f_2 \in F$ an element of F denoted $f_1 \cdot f_2$ (or sometimes $f_1 f_2$),

for which the following properties are satisfied:

1. for all $f_1, f_2, f_3 \in F$ we have $(f_1 + f_2) + f_3 = f_1 + (f_2 + f_3)$,
2. for all $f_1, f_2 \in F$ we have $f_1 + f_2 = f_2 + f_1$,
3. there is a designated element of F denoted by 0 for which $0 + f = f + 0 = f$ for all f,
4. for each $f \in F$, there exists $g \in F$, denoted $-f$ and called *the opposite of* f, such that $f+(-f) = (-f)+f = 0$,
5. for all $f_1, f_2, f_3 \in F$ we have $(f_1 f_2) f_3 = f_1 (f_2 f_3)$,
6. for all $f_1, f_2 \in F$ we have $f_1 f_2 = f_2 f_1$,
7. there is a designated element of F denoted by 1 for which $1 \cdot f = f \cdot 1 = f$ for all f,
8. for each $f \neq 0 \in F$, there exists $g \in F$, denoted f^{-1} and called *the inverse of* f, such that $ff^{-1} = f^{-1} f = 1$,
9. for all $f_1, f_2, f_3 \in F$, we have $f_1 \cdot (f_2 + f_3) = f_1 \cdot f_2 + f_1 \cdot f_3$,
10. $0 \neq 1$.

Example 3. The field of rational numbers $\mathbb{Q}$ consists of fractions with integer numerator and integer nonzero denominator (like $1/2$, $-5/3$, etc.).

Example 4. The field of real numbers $\mathbb{R}$ is our main example of a field; I assume that you know what it stands for.

Example 5. The field of complex numbers $\mathbb{C}$ consists, as you know, of expressions $a + bi$, where $a, b \in \mathbb{R}$ with obvious addition and multiplication that is completely defined by the rule $i^2 = -1$.

Example 6. An example which is absolutely foundational for computer science is the binary arithmetic: $\mathbb{F}_2 = \{0, 1\}$ with the operations $0 + 0 = 1 + 1 = 0$, $0 + 1 = 1 + 0 = 1$, $0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0$, $1 \cdot 1 = 1$.