1111: Linear Algebra I

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Lecture 13

Linear maps

A function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is called a *linear map* if two conditions are satisfied:

- for all $v_1, v_2 \in \mathbb{R}^n$, we have $f(v_1 + v_2) = f(v_1) + f(v_2)$;
- for all $v \in \mathbb{R}^n$ and all $c \in \mathbb{R}$, we have $f(c \cdot v) = c \cdot f(v)$.

Talking about matrix products, I suggested to view the product Ax as a function from $\mathbb{R}^n$ to $\mathbb{R}^m$. It turns out that all linear maps are like that.

Theorem. Let f be a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$. Then there exists a matrix A such that $f(x) = Ax$ for all x.

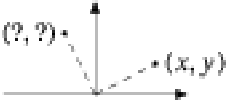
Proof. Let $\mathbf{e}_1, \ldots \mathbf{e}_\mathbf{n}$ be the standard unit vectors in $\mathbb{R}^n$: the vector $\mathbf{e}_i$ has its i-th coordinate equal to 1, and other coordinates equal to 0. Let $v_k = f(\mathbf{e}_k)$, and let us define a matrix A by putting together the vectors $v_1, \ldots, v_n$: $A = (v_1 \mid v_2 \mid \cdots \mid v_n)$. I claim that for every x we have $f(x) = Ax$. Indeed, we have

$$f(x) = f(x_1\mathbf{e_1} + \cdots + x_n\mathbf{e_n}) = x_1 f(\mathbf{e_1}) + \cdots + x_n f(\mathbf{e_n}) =$$
$$= x_1 A\mathbf{e}_1 + \cdots + x_n A\mathbf{e}_n = A(x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n) = Ax.$$

Linear maps: example

So far all maps that we considered were of the form $x \mapsto Ax$, so the result that we proved is not too surprising. Let me give an example of a linear map of geometric origin.

Let us consider the map that rotates every point counterclockwise through the angle 90° about the origin:



Since the standard unit vector $\mathbf{e}_1$ is mapped to $\mathbf{e}_2$, and $\mathbf{e}_2$ is mapped to $-\mathbf{e}_1$, the matrix that corresponds to this map is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. This means that each vector $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is mapped to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$. This can also be computed directly by inspection.

Linear independence, span, and linear maps

Let $v_1, \ldots, v_k$ be vectors in $\mathbb{R}^n$. Consider the $n \times k$-matrix A whose columns are these vectors.

Let us relate linear independence and the spanning property to linear maps. We shall now show that

- the vectors $v_1, \ldots, v_k$ are linearly independent if and only if the map from $\mathbb{R}^k$ to $\mathbb{R}^n$ that send each vector x to the vector Ax is *injective*, that is maps different vectors to different vectors;
- the vectors $v_1, \ldots, v_k$ span $\mathbb{R}^n$ if and only if the map from $\mathbb{R}^k$ to $\mathbb{R}^n$ that send each vector x to the vector Ax is *surjective*, that is something is mapped to every vector b in $\mathbb{R}^n$.

Indeed, we can note that injectivity means that $Ax = b$ has at most one solution for each b, which is equivalent to the absence of free variables, which is equivalent to the system $Ax = 0$ having only the trivial solution, which we know to be equivalent to linear independence.
Also, surjectivity means that $Ax = b$ has solutions for every b, which we know to be equivalent to the spanning property.

Subspaces of $\mathbb{R}^n$

A non-empty subset U of $\mathbb{R}^n$ is called a *subspace* if the following properties are satisfied:

- whenever $v, w \in U$, we have $v + w \in U$;
- whenever $v \in U$, we have $c \cdot v \in U$ for every scalar c.

Of course, this implies that every linear combination of several vectors in U is again in U.

Let us give some examples. Of course, there are two very trivial examples: $U = \mathbb{R}^n$ and $U = \{0\}$.
The line $y = x$ in $\mathbb{R}^2$ is another example.
Any line or 2D plane containing the origin in $\mathbb{R}^3$ would also give an example, and these give a general intuition of what the word "subspace" should make one think of.
The set of all vectors with integer coordinates in $\mathbb{R}^2$ is an example of a subset which is NOT a subspace: the first property is satisfied, but the second one certainly fails.

Subspaces of $\mathbb{R}^n$: two main examples

Let A be an $m \times n$-matrix. Then the solution set to the homogeneous system of linear equations $Ax = 0$ is a subspace of $\mathbb{R}^n$. Indeed, it is non-empty because it contains $x = 0$. We also see that if $Av = 0$ and $Aw = 0$, then $A(v + w) = Av + Aw = 0$, and similarly if $Av = 0$, then $A(c \cdot v) = c \cdot Av = 0$.

Let $v_1, \ldots, v_k$ be some given vectors in $\mathbb{R}^n$. Their linear span $\mathrm{span}(v_1, \ldots, v_k)$ is the set of all possible linear combinations $c_1 v_1 + \ldots + c_k v_k$. The linear span of $k \geqslant 1$ vectors is a subspace of $\mathbb{R}^n$. Indeed, it is manifestly non-empty, and closed under sums and scalar multiples.

The example of the line $y = x$ from the previous slide fits into both contexts. First of all, it is the solution set to the system of equations $A\mathbf{x} = 0$, where $A = \begin{pmatrix} 1 & -1 \end{pmatrix}$, and $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$. Second, it is the linear span of the vector $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. We shall see that it is a general phenomenon: these two descriptions are equivalent.

Subspaces of $\mathbb{R}^n$: two main examples

Consider the matrix $A = \begin{pmatrix} 1 & -2 & 1 & 0 \\ 3 & -5 & 3 & -1 \end{pmatrix}$, and the corresponding system of equations $Ax = 0$. The reduced row echelon form of this matrix is $\begin{pmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 0 & -1 \end{pmatrix}$, so the free unknowns are x_3 and x_4. Setting $x_3 = s$, $x_4 = t$, we obtain the solution $\begin{pmatrix} -s+2t \\ t \\ s \\ t \end{pmatrix}$, which we can represent as $s\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix}$. We conclude that the solution set to the system of equations is the linear span of the vectors $v_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix}$.

Subspaces of $\mathbb{R}^n$: two main examples

Let us implement this approach in general. Suppose A is an $m \times n$-matrix. As we know, to describe the solution set for $Ax = 0$ we bring A to its reduced row echelon form, and use free unknowns as parameters. Let x_{i_1}, ..., x_{i_k} be free unknowns. For each $j = 1, \ldots, k$, let us define the vector v_j to be the solution obtained by putting the j-th free unknown to be equal to 1, and all others to be equal to zero. Note that the solution that corresponds to arbitrary values $x_{i_1} = t_1$, ..., $x_{i_k} = t_k$ is the linear combination $t_1 v_1 + \cdots + t_k v_k$. Therefore the solution set of $Ax = 0$ is the linear span of v_1, ..., v_k.

Note that in fact the vectors v_1, ..., v_k constructed above are linearly independent. Indeed, the linear combination $t_1 v_1 + \cdots + t_k v_k$ has t_i in the place of i-th free unknown, so if this combination is equal to zero, then all coefficients must be equal to zero. Therefore, it is sensible to say that these vectors form a basis in the solution set: every vector can be obtained as their linear combination, and they are linearly independent. However, we only considered bases of $\mathbb{R}^n$ so far, and the solution set of a system of linear equations differs from $\mathbb{R}^m$.