

1111: LINEAR ALGEBRA I

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Lecture 12

DETERMINANTS AND CROSS PRODUCTS

Reminder: vector product (cross product) of two vectors \mathbf{v} and \mathbf{w} . If \mathbf{v} has coordinates a, b, c and \mathbf{w} has coordinates a_1, b_1, c_1 , then, by definition, $\mathbf{v} \times \mathbf{w}$ has coordinates

$$bc_1 - b_1c, ca_1 - c_1a, ab_1 - a_1b .$$

Let us write this using standard unit vectors:

$$\mathbf{v} \times \mathbf{w} = (bc_1 - b_1c)\mathbf{i} + (ca_1 - c_1a)\mathbf{j} + (ab_1 - a_1b)\mathbf{k}.$$

Formally, this can be written as the first column expansion of the determinant

$$\det \begin{pmatrix} \mathbf{i} & a & a_1 \\ \mathbf{j} & b & b_1 \\ \mathbf{k} & c & c_1 \end{pmatrix},$$

if we momentarily forget that \mathbf{i} , \mathbf{j} , and \mathbf{k} are vectors, not numbers, and hence cannot / should not be used as entries of a matrix.

DETERMINANTS AND VOLUMES

The formula

$$\mathbf{v} \times \mathbf{w} = \det \begin{pmatrix} \mathbf{i} & a & a_1 \\ \mathbf{j} & b & b_1 \\ \mathbf{k} & c & c_1 \end{pmatrix}$$

explains also why the volume of the parallelepiped determined by \mathbf{u} , \mathbf{v} , and \mathbf{w} is equal to the absolute value of the determinant of the matrix obtained by putting these vectors together.

This follows from the fact that the latter volume is equal to

$$|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|,$$

which can now be viewed as an honest determinant expansion.

In higher dimensions, determinants are used to define volumes, and to compute volumes of curved shapes by integration; you shall see a lot of that in your analysis / calculus modules.

THE COORDINATE VECTOR SPACE \mathbb{R}^n

We already used vectors in n dimensions when talking about systems of linear equations. However, we shall now introduce some further notions and see how those notions may be applied.

Recall that the coordinate vector space \mathbb{R}^n consists of all columns of height n with real entries, which we refer to as *vectors*.

Let v_1, \dots, v_k be vectors, and let c_1, \dots, c_k be real numbers. The *linear combination* of vectors v_1, \dots, v_k with coefficients c_1, \dots, c_k is, quite unsurprisingly, the vector $c_1 v_1 + \dots + c_k v_k$.

The vectors v_1, \dots, v_k are said to be *linearly independent* if the only linear combination of these vectors which is equal to the zero vector is the combination where all coefficients are equal to 0. Otherwise those vectors are said to be *linearly dependent*.

The vectors v_1, \dots, v_k are said to *span* \mathbb{R}^n , or to form a *complete set of vectors*, if every vector can be written as some linear combination of those vectors.

LINEAR INDEPENDENCE AND SPAN: EXAMPLES

If a system of vectors contains the zero vector, these vectors may not be linearly independent, since it is enough to take the zero vector with a nonzero coefficient.

If a system of vectors contains two equal vectors, or two proportional vectors, these vectors may not be linearly independent. More generally, several vectors are linearly dependent if and only if one of those vectors can be represented as a linear combination of others. (*Exercise: prove that last statement*).

The standard unit vectors e_1, \dots, e_n are linearly independent; they also span \mathbb{R}^n .

If the given vectors are linearly independent, then removing some of them keeps them linearly independent. If the given vectors span \mathbb{R}^n , then throwing in some extra vectors does not destroy this property.

LINEAR COMBINATIONS AND SYSTEMS OF LINEAR EQUATIONS

Let us make one very important observation:

For an $n \times k$ -matrix A and a vector x of height k , the product Ax is the linear combination of columns of A whose coefficients

are the coordinates of the vector x . If $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix}$, and

$A = (v_1 \mid v_2 \mid \cdots \mid v_k)$, then $Ax = x_1 v_1 + \cdots + x_k v_k$.

We already utilised that when working with systems of linear equations.

LINEAR INDEPENDENCE AND SPAN

Let v_1, \dots, v_k be vectors in \mathbb{R}^n . Consider the $n \times k$ -matrix A whose columns are these vectors.

Clearly, the vectors v_1, \dots, v_k are linearly independent if and only if the system of equations $Ax = 0$ has only the trivial solution. This happens if and only if there are no free variables, so the reduced row echelon form of A has a pivot in every column.

Clearly, the vectors v_1, \dots, v_k span \mathbb{R}^n if and only if the system of equations $Ax = b$ has solutions for every b . This happens if and only if the reduced row echelon form of A has a pivot in every row. (Indeed, otherwise for some b we shall have the equation $0 = 1$).

In particular, if v_1, \dots, v_k are linearly independent in \mathbb{R}^n , then $k \leq n$ (there is a pivot in every column of A , and at most one pivot in every row), and if v_1, \dots, v_k span \mathbb{R}^n , then $k \geq n$ (there is a pivot in every row, and at most one pivot in every column).

BASES OF \mathbb{R}^n

We say that vectors v_1, \dots, v_k in \mathbb{R}^n form a *basis* if they are linearly independent and they span \mathbb{R}^n .

Theorem. Every basis of \mathbb{R}^n consists of exactly n elements.

Proof. We know that if v_1, \dots, v_k are linearly independent, then $k \leq n$, and if v_1, \dots, v_k span \mathbb{R}^n , then $k \geq n$. Since both properties are satisfied, we must have $k = n$.

Let v_1, \dots, v_n be vectors in \mathbb{R}^n . Consider the $n \times n$ -matrix A whose columns are these vectors. Our previous results immediately show that v_1, \dots, v_n form a basis if and only if the matrix A is invertible (for which we had many equivalent conditions last week).