1111: Linear Algebra I

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Row expansion of the determinant

Our next goal is to prove the following formula:

$$\det(A) = a_{i1}C^{i1} + a_{i2}C^{i2} + \cdots + a_{in}C^{in},$$

in other words, if we multiply each element of the i-th row of A by its cofactor and add results, the number we obtain is equal to $\det(A)$.

Let us first handle the case $i = 1$. In this case, once we write the corresponding minors with signs, the formula reads

$$\det(A) = a_{11}A^{11} - a_{12}A^{12} + \cdots + (-1)^{n+1}a_{1n}A^{1n}.$$

Let us examine the formula for $\det(A)$. It is a sum of terms corresponding to pick n elements representing each row and each column of A. In particular, it involves picking an element from the first row. This can be one of the elements $a_{11}, a_{12}, \ldots, a_{1n}$. What remains, after we picked the element a_{1k}, is to pick other elements; now we have to avoid the row 1 and the column k. This means that the elements we need to pick are precisely those involved in the determinant A^{1k}, and we just need to check that the signs match.

Row expansion of the determinant

In fact, it is quite easy to keep track of all signs. The column $\begin{pmatrix}1\\k\end{pmatrix}$ in the two-row notation does not add inversions in the first row, and adds $k-1$ inversions in the second row, since k appears before $1, 2, \ldots, k-1$. This shows that the signs indeed have that mismatch $(-1)^{k-1} = (-1)^{k+1}$, as we claim.

Note that the case of arbitrary i is not difficult. We can just reduce it to the case of $i = 1$ by performing $i - 1$ row swaps. This multiplies the determinant by $(-1)^{i-1}$, which matches precisely the signs in cofactors in the formula

$$\det(A) = a_{i1}C^{i1} + a_{i2}C^{i2} + \cdots + a_{in}C^{in} \ .$$

“Wrong row expansion”

In fact, another similar formula also holds: for $i \neq j$, we have

$$a_{i1}C^{j1} + a_{i2}C^{j2} + \cdots + a_{in}C^{jn} = 0 \; .$$

It follows instantly from what we already proved: take the matrix A' which is obtained from A by replacing the j-th row by a copy of the i-th row. Then the left hand side is just the j-th row expansion of $\det(A')$, and it remains to notice that $\det(A') = 0$ because this matrix has two equal rows.

These results altogether can be written like this:

$$a_{i1}C^{j1} + a_{i2}C^{j2} + \cdots + a_{in}C^{jn} = \det(A)\delta_i^j \; .$$

Here δ_i^j is the *Kronecker symbol*; it is equal to 1 for $i = j$ and is equal to zero otherwise. In matrix notation, $A \cdot C^T = \det(A) \cdot I_n$. (Note that here we have a product of matrices in the firstcase, and the product of the matrix I_n and the scalar $\det(A)$ in the second case).

Adjugate matrix

The transpose of cofactor matrix $C = (C^{ij})$ is called the *adjugate matrix* of the matrix A, and is denoted $\mathrm{adj}(A)$. (Historically it was called the adjoint matrix, but now that term is used for something else). For example, for the matrix $A = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 1 & -2 \\ 0 & 1 & 1 \end{pmatrix}$ we have

$C = \begin{pmatrix} 3 & -2 & 2 \\ -3 & 1 & -1 \\ -6 & 2 & -5 \end{pmatrix}$, and $\mathrm{adj}(A) = \begin{pmatrix} 3 & -3 & -6 \\ -2 & 1 & 2 \\ 2 & -1 & -5 \end{pmatrix}$.

We already proved one half of the following result:

Theorem. For each $n \times n$-matrix A, we have

$$A \cdot \mathrm{adj}(A) = \mathrm{adj}(A) \cdot A = \det(A) \cdot I_n.$$

Adjugate matrix

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The other half is proved by taking transposes: from what we already proved, we have $A^T \cdot C = \det(A^T) \cdot I_n$, because the cofactor matrix of A^T is C^T. Now, taking transposes, and using $\det(A^T) = \det(A)$, we see that $C^T \cdot A = (A^T \cdot C)^T = (\det(A) \cdot I_n)^T = \det(A) \cdot I_n$.

Similarly to how the first half of this theorem encodes row expansion for determinants, the second half encodes the similar *column expansion*: if you multiply each element of the i-th column by its cofactor and add results, you get the determinant.

A closed formula for the inverse matrix

Theorem. Suppose that $\det(A) \neq 0$. Then

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A).$$

Proof. Indeed, take the formula $A \cdot \operatorname{adj}(A) = \operatorname{adj}(A) \cdot A = \det(A) \cdot I_n$, and divide by $\det(A)$. □

This theorem shows that not only a matrix is invertible when the determinant is not equal to zero, but also that you can compute the inverse by doing exactly one division; all other operations are addition, subtraction, and multiplication.

Cramer's formula for systems of linear equations

We know that if A is invertible then $Ax = b$ has just one solution $x = A^{-1}b$. Let us plug in the formula for A^{-1} that we have:

$$x = A^{-1}b = \frac{1}{\det(A)}\,\mathrm{adj}(A)b\ .$$

When we compute $\mathrm{adj}(A)b = C^T b$, we get the vector whose k-th entry is

$$C_{1k}b_1 + C_{2k}b_2 + \ldots + C_{nk}b_n\ .$$

What does it look like? It looks like a k-th column expansion of some determinant, more precisely, of the determinant of the matrix A_k which is obtained from A by replacing its k-th column with b. (This way, the cofactors of that column do not change).

Theorem. (Cramer's formula) Suppose that $\det(A) \neq 0$. Then coordinates of the only solution to the system of equations $Ax = b$ are

$$x_k = \frac{\det(A_k)}{\det(A)}\ .$$

Summary of systems of linear equations

Theorem. Let us consider a system of linear equations $Ax = b$ with n equations and n unknowns. The following statements are equivalent:

(a) the *homogeneous* system $Ax = 0$ has only the trivial solution $x = 0$;
(b) the reduced row echelon form of A is I_n;
(c) $\det(A) \neq 0$.
(d) the matrix A is invertible;
(e) the system $Ax = b$ has exactly one solution;

Proof. In principle, to show that five statements are equivalent, we need to do a lot of work. We could, for each pair, prove that they are equivalent, altogether $5 \cdot 4 = 20$ proofs. We could prove that $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d) \Leftrightarrow (e)$, altogether 8 proofs. What we shall do instead is prove $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (a)$, just 5 proofs.

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Proof. $(a) \Rightarrow (b)$: by contradiction, if the reduced row echelon form has a row of zeros, we get free variables.
$(b) \Rightarrow (c)$: follows from properties of determinants, elementary operations multiply the determinant by nonzero scalars.
$(c) \Rightarrow (d)$: proved in several different ways already.
$(d) \Rightarrow (e)$: discussed early on, if A is invertible, then $x = A^{-1}b$ is clearly the only solution to $Ax = b$.
$(e) \Rightarrow (a)$: by contradiction, if v a solution to $Ax = b$ and w is a nontrivial solution to $Ay = 0$, then $v + w$ is another solution to $Ax = b$.