MA 1111: Linear Algebra I
Selected answers/solutions to the assignment due December 10, 2015

1. (**a**) Solving the systems of equations $f_1 = c_{11}e_1+c_{21}e_2+c_{31}e_3$, $f_2 = c_{12}e_1+c_{22}e_2+c_{32}e_3$, $f_3 = c_{13}e_1 + c_{23}e_2 + c_{33}e_3$, we get $c_{11} = -3/2$, $c_{21} = 5/2$, $c_{31} = 1/2$, $c_{12} = -1$, $c_{22} = 1$, $c_{32} = 0$, $c_{13} = 1/2$, $c_{23} = -3/2$, $c_{33} = 3/2$. Therefore, $M_{\mathbf{e},\mathbf{f}} = \begin{pmatrix} -3/2 & -1 & 1/2 \\ 5/2 & 1 & -3/2 \\ 1/2 & 0 & 3/2 \end{pmatrix}$.

(**b**) By a result proved in class, the column of those coordinates is

$$M_{\mathbf{e},\mathbf{f}} \begin{pmatrix} 1 \\ 7 \\ -3 \end{pmatrix} = \begin{pmatrix} -10 \\ 14 \\ -4 \end{pmatrix}.$$

2. Observe that $1 = 1 \cdot 1$, $t + 1 = 1 \cdot 1 + 1 \cdot t$, $(t + 1)^2 = 1 \cdot 1 + 2 \cdot t + 1 \cdot t^2$, and $(t + 1)^3 = 1 \cdot 1 + 3 \cdot t + 3 \cdot t^2 + 1 \cdot t^3$. Therefore, the transition matrices are

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

3. (a) and (b) are linear operators; by direct inspection, sums are mapped to sums, and scalar multiples are mapped to scalar multiples. (c) is not a linear operator; $f(t) = t^2$ is mapped to $2 \cdot 2t = 4t$, and $-t^2$ is mapped to $(-2) \cdot (-2t) = 4t$ as well, even though its image should be the opposite vector.

4. (**a**) We have $v \cdot (w_1 + w_2) = v \cdot w_1 + v \cdot w_2$ and $v \cdot (cw) = cv \cdot w$ by the known properties of dot products, so the operator is linear. Also, $v \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1$, $v \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 2$, $v \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = -1$, so the matrix of our operator is $\begin{pmatrix} 1 & 2 & -1 \end{pmatrix}$.

(**b**) We have $v \times (w_1 + w_2) = v \times w_1 + v \times w_2$ and $v \times (cw) = cv \times w$ by the known properties of cross products, so the operator is linear. Also, $v \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -2 \end{pmatrix}$, $v \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $v \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$, so the matrix of our operator is $\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix}$.

5. Let us compute the transition matrix $M_{\mathbf{e},\mathbf{f}}$. Solving the systems of equations $f_1 = c_{11}e_1 + c_{21}e_2$ and $f_2 = c_{12}e_1 + c_{22}e_2$, we get $c_{11} = -87$, $c_{21} = 25$, $c_{12} = -7$, $c_{22} = 2$, therefore $M_{\mathbf{e},\mathbf{f}} = \begin{pmatrix} -87 & -7 \\ 25 & 2 \end{pmatrix}$. Therefore, we have

$$A_{\varphi,\mathbf{f}} = M_{\mathbf{e},\mathbf{f}}^{-1} A_{\varphi,\mathbf{e}} M_{\mathbf{e},\mathbf{f}} = \begin{pmatrix} -1442 & -116 \\ 17938 & 1443 \end{pmatrix}.$$