MA 1111: Linear Algebra I

Selected answers/solutions to the assignment due November 19, 2015

1. (**a**) The matrix $A = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$ is clearly invertible (its determinant is equal to $-5 \neq 0$), so its reduced row echelon form is I_2, that reduced row echelon form has a pivot in each row and each column, therefore these vectors span $\mathbb{R}^2$, are linearly independent, and form a basis.

(**b**) The reduced row echelon form of the matrix $A = \begin{pmatrix} -1 & 2 & 7 \\ 2 & 1 & 6 \end{pmatrix}$ is $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \end{pmatrix}$; it has no zero rows, so these vectors span $\mathbb{R}^2$; it has one column without a pivot, so these vectors are linearly dependent. Consequently, they do not form a basis.

2. The reduced row echelon form of the matrix $A = \begin{pmatrix} -1 & 2 & 7 \\ 2 & 1 & 6 \\ 0 & 1 & -1 \end{pmatrix}$ is I_3. Since each row of I_3 has a pivot, these vectors span $\mathbb{R}^3$, and since each column of I_3 has a pivot, these vectors are linearly independent. Therefore, they form a basis.

3. The reduced row echelon form of the matrix $A = \begin{pmatrix} -1 & 2 & 7 \\ 2 & 1 & 6 \\ 1 & 0 & 1 \end{pmatrix}$ is $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}$. Since the reduced row echelon form has a row with no pivot, these vectors do not span $\mathbb{R}^3$, and since there is a column with no pivot, these vectors are not linearly independent. Therefore, of course, they do not form a basis.

4. The determinant of the matrix formed by these vectors is the Vandermonde determinant and hence is not equal to zero. Therefore, the reduced row echelon form of that matrix is I_n, so these vectors are linearly independent, span $\mathbb{R}^n$, and form a basis.

5. Suppose that

$$c_1(u - 2w) + c_2(v + w) + c_3 w = 0$$

for some coefficients c_1, c_2, c_3. This can be rewritten as

$$c_1 u + c_2 v + (-2c_1 + c_2 + c_3)w = 0.$$

Since the vectors u, v, w are linearly independent, we conclude that $c_1 = 0$, $c_2 = 0$, and $-2c_1 + c_2 + c_3 = 0$, which implies $c_1 = c_2 = c_3 = 0$. Hence, the vectors $u - 2w$, $v + w$, and w are linearly independent.