1. $\Delta_1 = 3, \Delta_2 = 2, \Delta_3 = -7$, so by Jacobi’s theorem the signature can be read from the sequence $1/3, 3/2, -2/7$; it is $(2, 1, 0)$.
2. The matrix of the corresponding bilinear form is

$$A = \begin{pmatrix}
18 + a & 5 & -a - 4 \\
5 & 3 & -2 \\
-a - 4 & -2 & a
\end{pmatrix}.$$ 

We have $\Delta_1 = 18 + a, \Delta_2 = 3a + 29, \Delta_3 = 21a - 40$. All these numbers are positive if and only if

$$a > \frac{40}{21}.$$ 

3. The characteristic polynomial of the matrix $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ is $2t - t^3 = t(2 - t^2)$, so the eigenvalues of this matrix are 0 and $\pm \sqrt{2}$. The matrix of the bilinear form corresponding to the quadratic form

$$q(x_1e_1 + x_2e_2 + x_3e_3) = x_1x_2 + x_2x_3$$

is equal to $1/2$ of the matrix in question, so its signature can be read off the eigenvalues of this matrix, and is $(1, 1, 1)$.

4. We have

$$\frac{\partial \varphi}{\partial x_1} = 2\sin(x_1 - x_2) \cos(x_1 - x_2) - (x_1 + 2cx_2)e^{\frac{1}{2}(x_1^2 + x_2^2) + 2cx_1x_2} =$$

$$= \sin 2(x_1 - x_2) - (x_1 + 2cx_2)e^{\frac{1}{2}(x_1^2 + x_2^2) + 2cx_1x_2},$$

and

$$\frac{\partial \varphi}{\partial x_2} = -2\sin(x_1 - x_2) \cos(x_1 - x_2) - (2cx_1 + x_2)e^{\frac{1}{2}(x_1^2 + x_2^2) + 2cx_1x_2} =$$

$$= -\sin 2(x_1 - x_2) - (2cx_1 + x_2)e^{\frac{1}{2}(x_1^2 + x_2^2) + 2cx_1x_2},$$

and

$$\frac{\partial^2 \varphi}{\partial x_1^2} = 2\cos 2(x_1 - x_2) - e^{\frac{1}{2}(x_1^2 + x_2^2) + 2cx_1x_2} - (x_1 + 2cx_2)^2 e^{\frac{1}{2}(x_1^2 + x_2^2) + 2cx_1x_2},$$

$$\frac{\partial^2 \varphi}{\partial x_1 \partial x_2} = -2\cos 2(x_1 - x_2) - 2ce^{\frac{1}{2}(x_1^2 + x_2^2) + 2cx_1x_2} - (x_1 + 2cx_2)(2cx_1 + x_2)e^{\frac{1}{2}(x_1^2 + x_2^2) + 2cx_1x_2},$$

$$\frac{\partial^2 \varphi}{\partial x_2^2} = 2\cos 2(x_1 - x_2) - e^{\frac{1}{2}(x_1^2 + x_2^2) + 2cx_1x_2} - (2cx_1 + x_2)^2 e^{\frac{1}{2}(x_1^2 + x_2^2) + 2cx_1x_2},$$
so the matrix $A$ is
\[
\begin{pmatrix}
1 & -2 - 2c \\
-2 - 2c & 1
\end{pmatrix}.
\]

By Sylvester’s criterion, this quadratic form is positive definite if and only if $\Delta_2 = 1 - (2 + 2c)^2 > 0$ (since $\Delta_1 = 1$). We have
\[
1 - (2 + 2c)^2 = (1 + 2 + 2c)(1 - 2 - 2c) = (3 + 2c)(-1 - 2c),
\]
so the quadratic form is positive definite for $-3/2 < c < -1/2$. In particular, this holds for $c = -3/5$. 