MA 1212: Linear Algebra II
Tutorial problems, January 29, 2015

1. First we make this set into a set of orthogonal vectors. We put

$$e_1 = f_1 = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix},$$

$$e_2 = f_2 - \frac{(e_1, f_2)}{(e_1, e_1)} e_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix},$$

$$e_3 = f_3 - \frac{(e_1, f_3)}{(e_1, e_1)} e_1 - \frac{(e_2, f_3)}{(e_2, e_2)} e_2 = \begin{pmatrix} 12/19 \\ -2/19 \\ -2/19 \end{pmatrix}.$$

To conclude, we normalise the vectors, obtaining the answer

$$\frac{1}{\sqrt{19}} \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad \frac{1}{\sqrt{38}} \begin{pmatrix} 6 \\ -1 \\ -1 \end{pmatrix}.$$

2. This formula is bilinear and symmetric by inspection. Also, if we put $x_1 = x_2$ and $y_1 = y_2$, we obtain $x_1^2 + x_1 y_1 + y_1^2 = (x_1 + \frac{1}{2} y_1)^2 + \frac{3}{4} y_1^2$, and we see that this can only be equal to zero for $x_1 = y_1 = 0$, so the positivity holds as well. Let us apply the Gram–Schmidt process to the standard unit vectors. This means that we would like to replace e_2 by $e_2 - \frac{(e_1, e_2)}{(e_1, e_1)} e_1 = \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$. It remains to normalise these vectors, obtaining

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -1/\sqrt{3} \\ 2\sqrt{3} \end{pmatrix}$$

3. We first orthogonalise these vectors, noting that $\int_{-1}^{1} f(t)\, dt$ is equal to 0 if $f(t)$ is an odd function (this shows that our computations are actually quite easy, because even powers of t are automatically orthogonal to odd powers):

$$
\begin{aligned}
e_1 &= 1, \\
e_2 &= t - \frac{(1,t)}{(1,1)}1 = t, \\
e_3 &= t^2 - \frac{(1,t^2)}{(1,1)}1 - \frac{(t,t^2)}{(t,t)}t = t^2 - \frac{1}{3}, \\
e_4 &= t^3 - \frac{(1,t^3)}{(1,1)}1 - \frac{(t,t^3)}{(t,t)}t - \frac{(t^2-\frac{1}{3},t^3)}{(t^2-\frac{1}{3},t^2-\frac{1}{3})}(t^2-\frac{1}{3}) = t^3 - \frac{3}{5}.
\end{aligned}
$$

To conclude, we normalise these vectors, obtaining

$$
\frac{1}{\sqrt{2}}, \frac{\sqrt{3}t}{\sqrt{2}}, \frac{\sqrt{5}(3t^2-1)}{2\sqrt{2}}, \frac{\sqrt{7}(5t^3-3t)}{2\sqrt{2}}.
$$