1212: Linear Algebra II

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Lecture 4

One more example for diagonalisation

Let V is the space of all polynomials in t of degree at most n, and let $\varphi\colon V \to V$ be the linear transformation given by $(\varphi(p(t)) = p(t) - 2p'(t)$. Let us find out whether φ can be diagonalised.

If $\varphi(p(t)) = \lambda p(t)$, we have $p(t) - 2p'(t) = \lambda p(t)$, or $(1-\lambda)p(t) - 2p'(t) = 0$. If $\lambda \neq 1$, the leading term of $p(t)$ will not cancel, so there are no nontrivial solutions. If $\lambda = 1$, we have $-2p'(t) = 0$, so $p(t)$ is a constant. Clearly, for $n > 0$ there is basis consisting of eigenvectors (we cannot express polynomials of positive degree using eigenvectors), so the matrix of φ cannot be made diagonal by the change of basis.

All the same can be done using the basis 1, t, ..., t^n of the space of polynomials; the matrix of our operator relative to this basis is, as it is easy to see,

$$\begin{pmatrix} 1 & -2 & 0 & \dots & \dots & 0 \\ 0 & 1 & -4 & \dots & \dots & 0 \\ 0 & 0 & 1 & \dots & \dots & 0 \\ \vdots & \vdots & \dots & \ddots & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 & -2n \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix},$$

and all our statements easily follow from computations with matrices.

Sums and direct sums

Let V be a vector space. Recall that the *span* of a set of vectors $v_1, \dots, v_k \in V$ is the set of all linear combinations $c_1v_1 + \dots + c_kv_k$. It is denoted by $\text{span}(v_1, \dots, v_k)$. Vectors $v_1, \dots, v_k$ are linearly independent if and only if they form a basis of their linear span. Our next definition provides a generalization of what is just said, dealing with subspaces, and not vectors.

Definition 1. Let V_1, ..., V_k be subspaces of V. Their *sum* $V_1 + \dots + V_k$ is defined as the set of vectors of the form $v_1 + \dots + v_k$, where $v_1 \in V_1$, ..., $v_k \in V_k$. The sum of the subspaces V_1, ..., V_k is said to be *direct* if $0 + \dots + 0$ is the only way to represent $0 \in V_1 + \dots + V_k$ as a sum $v_1 + \dots + v_k$. In this case, it is denoted by $V_1 \oplus \dots \oplus V_k$.

Lemma 1. $V_1 + \dots + V_k$ *is a subspace of* V.

Proof. It is sufficient to check that $V_1 + \dots + V_k$ is closed under addition and multiplication by numbers. Clearly,

$$(v_1 + \dots + v_k) + (v_1' + \dots + v_k') = ((v_1 + v_1') + \dots + (v_k + v_k'))$$

and

$$c(v_1 + \dots + v_k) = ((cv_1) + \dots + (cv_k)),$$

and the lemma follows, since each V_i is a subspace and hence closed under the vector space operations. □

Example 1. Let $V = \mathbb{R}^3$, and let $V_1 = \text{span}(e_1, e_2)$, $V_2 = \text{span}(e_2, e_3)$, where e_i are standard unit vectors. Then the sum $V_1 + V_2$ consists of all combinations of e_1, e_2, e_3, so $V_1 + V_2 = V$. This sum is not direct, since $0 = e_2 - e_2$ is a nontrivial representation of 0.

On the other hand, for $V_1 = \text{span}(e_1, e_2)$ and $V_2 = \text{span}(e_3)$ the sum is still equal to V and is direct (exercise).

Example 2. For a collection of nonzero vectors $v_1, \ldots, v_k \in V$, consider the subspaces $V_1, \ldots, V_k$, where V_i consists of all multiples of v_i. Then, clearly, $V_1 + \ldots + V_k = \text{span}(v_1, \ldots, v_k)$, and this sum is direct if and only if the vectors v_j are linearly independent.

Example 3. For two subspaces V_1 and V_2, their sum is direct if and only if $V_1 \cap V_2 = \{0\}$. Indeed, if $v_1 + v_2 = 0$ is a nontrivial representation of 0, $v_1 = -v_2$ is in the intersection, and vice versa.

Theorem 1. *If V_1 and V_2 are subspaces of V, we have*

$$\dim(V_1 + V_2) = \dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2).$$

In particular, the sum of V_1 and V_2 is direct if and only if $\dim(V_1 + V_2) = \dim(V_1) + \dim(V_2)$.

Proof. Let us pick a basis $e_1, \ldots, e_k$ of the intersection $V_1 \cap V_2$, and extend this basis to a bigger set of vectors in two different ways, one way obtaining a basis of V_1, and the other way — a basis of V_2. Let $e_1, \ldots, e_k, f_1, \ldots, f_l$ and $e_1, \ldots, e_k, g_1, \ldots, g_m$ be the resulting bases of V_1 and V_2 respectively. Let us prove that

$$e_1, \ldots, e_k, f_1, \ldots, f_l, g_1, \ldots, g_m$$

is a basis of $V_1 + V_2$. It is a complete system of vectors, since every vector in $V_1 + V_2$ is a sum of a vector from V_1 and a vector from V_2, and vectors there can be represented as linear combinations of $e_1, \ldots, e_k, f_1, \ldots, f_l$ and $e_1, \ldots, e_k, g_1, \ldots, g_m$ respectively. To prove linear independence, let us assume that

$$a_1 e_1 + \ldots + a_k e_k + b_1 f_1 + \ldots + b_l f_l + c_1 g_1 + \ldots + c_m g_m = 0.$$

Rewriting this formula as $a_1 e_1 + \ldots + a_k e_k + b_1 f_1 + \ldots + b_l f_l = -(c_1 g_1 + \ldots + c_m g_m)$, we notice that on the left we have a vector from V_1 and on the right a vector from V_2, so both the left hand side and the right hand side is a vector from $V_1 \cap V_2$, and so can be represented as a linear combination of $e_1, \ldots, e_k$ alone. However, the vectors on the right hand side together with e_i form a basis of V_2, so there is no nontrivial linear combination of these vectors that is equal to a linear combination of e_i. Consequently, all coefficients c_i are equal to zero, so the left hand side is zero. This forces all coefficients a_i and b_i to be equal to zero, since $e_1, \ldots, e_k, f_1, \ldots, f_l$ is a basis of V_1. This completes the proof of the linear independence of the vectors $e_1, \ldots, e_k, f_1, \ldots, f_l, g_1, \ldots, g_m$.

Summing up, $\dim(V_1) = k + l$, $\dim(V_2) = k + m$, $\dim(V_1 + V_2) = k + l + m$, $\dim(V_1 \cap V_2) = k$, and our theorem follows. □

In practice, it is important sometimes to determine the intersection of two subspaces, each presented as a linear span of several vectors. We shall discuss it in the next class.