1212: Linear Algebra II

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Lecture 20

One further example

Example 1. $V = \mathbb{R}^4$, φ is multiplication by the matrix $A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & -1 \end{pmatrix}$. In this case, $\varphi^2 = 0$, $\text{rk}(\varphi) = 2$, $\text{rk}(\varphi^k) = 0$ for $k \geqslant 2$, $\text{null}(\varphi) = 2$, $\text{null}(\varphi^k) = 4$ for $k \geqslant 2$. Moreover, $\text{Ker}(\varphi) = \{\begin{pmatrix} -s \\ t \\ t \\ s \end{pmatrix}\}$. We have a sequence of subspaces $V = \text{Ker}(\varphi^2) \supset \text{Ker}(\varphi) \supset \{0\}$. The first one relative to the second one is two-dimensional ($\dim \text{Ker}(\varphi^2) - \dim \text{Ker}(\varphi) = 2$). Clearly, the vectors $\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ (corresponding to $s = 1, t = 0$ and $s = 0, t = 1$ respectively) form a basis of the kernel of φ, and after computing the reduced column echelon form and looking for missing pivots, we obtain a relative basis consisting of the vectors $f_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $f_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$. These vectors give rise to threads f_1, $\varphi(f_1) = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ and f_2, $\varphi(f_2) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$. These two threads together contain four vectors, and we have a basis.

Uniqueness of the normal form

Let us denote by m_d the number of threads of length d, where $1 \leqslant d \leqslant k$. In that case, we have

$$\begin{aligned} m_k &= \dim N_k - \dim N_{k-1}, \\ m_{k-1} + m_k &= \dim N_{k-1} - \dim N_{k-2}, \\ &\dots \\ m_2 + \ldots + m_k &= \dim N_2 - \dim N_1, \\ m_1 + m_2 + \ldots + m_k &= \dim N_1, \end{aligned}$$

so the numbers of threads of various lengths are uniquely determined by the characteristics of the linear transformation φ that do not depend on any choices (dimensions of kernels of powers).

Finding a direct sum decomposition

Now, suppose that φ is an arbitrary linear transformation of V. Consider the sequence of subspaces $N_1 = \ker(\varphi)$, $N_2 = \ker(\varphi^2)$, ..., $N_m = \ker(\varphi^m)$, ...

Note that this sequence is increasing:

$$N_1 \subset N_2 \subset \ldots N_m \subset \ldots$$

Indeed, if $v \in N_s$, that is $\varphi^s(v) = 0$, then we have

$$\varphi^{s+1}(v) = \varphi(\varphi^s(v)) = 0.$$

Since we only work with finite-dimensional vector spaces, this sequence of subspaces cannot be strictly increasing; if $N_i \neq N_{i+1}$, then, obviously, $\dim N_{i+1} \geqslant 1 + \dim N_i$. It follows that for some k we have $N_k = N_{k+1}$.

Lemma 1. *In this case we have* $N_{k+l} = N_k$ *for all* $l > 0$.

Proof. We shall prove that $N_{k+l} = N_{k+l-1}$ by induction on l. The induction basis (case $l = 1$) follows immediately from our notation. Suppose that $N_{k+l} = N_{k+l-1}$; let us prove that $N_{k+l+1} = N_{k+l}$. Let us take a vector $v \in N_{k+l+1}$, so $\varphi^{k+l+1}(v) = 0$. We have $\varphi^{k+l+1}(v) = \varphi^{k+l}(\varphi(v))$, so $\varphi(v) \in N_{k+l}$. But by the induction hypothesis $N_{k+l} = N_{k+l-1}$, so $\varphi^{k+l-1}(\varphi(v)) = 0$, or $\varphi^{k+l}(v) = 0$, so $v \in N_{k+l}$, as required. □

Lemma 2. *Under our assumptions, we have* $\ker(\varphi^k) \cap \operatorname{Im}(\varphi^k) = \{0\}$.

Proof. Indeed, suppose that $v \in \ker(\varphi^k) \cap \operatorname{Im}(\varphi^k)$. This means that $\varphi^k(v) = 0$ and that $v = \varphi^k(w)$ for some vector w. It follows that $\varphi^{2k}(w) = 0$, so $w \in N_{2k}$. But from the previous lemma we know that $N_{2k} = N_k$, so $w \in N_k$. Thus, $v = \varphi^k(w) = 0$, which completes the proof. □

Lemma 3. *Under our assumptions, we have* $V = \ker(\varphi^k) \oplus \operatorname{Im}(\varphi^k)$.

Proof. Indeed, consider the sum of these two subspaces (which is, as we just proved in the previous lemma, direct). It is a subspace of V of dimension $\dim \ker(\varphi^k) + \dim \operatorname{Im}(\varphi^k) = \dim(V)$, so it has to coincide with V. □