1212: Linear Algebra II

Dr. Vladimir Dotsenko (Vlad)

Lecture 18

Computing relative bases

Let us begin with a general remark on relative bases.

To compute a basis of $\mathbb{R}^n$ relative to the linear span of several vectors, one may compute the reduced column echelon form for the matrix made of those vectors, and pick, for a relative basis, those standard unit vectors corresponding to "missing leading 1's", that is to the non-principal rows of the reduced column echelon form.

More generally, if we are required to determine a basis of a vector space V relative to its subspace U, we can proceed as follows. Let A be a matrix whose columns form a basis of U, B — a matrix whose columns form a basis of V. We can find the reduced column echelon form R for A. Write R next to B, and use it to "reduce" B, making sure that all rows that contain pivots of R do not contain any other nonzero elements. Then it remains to find the reduced column echelon form of the matrix B' that replaces the matrix B. Its nonzero columns form a relative basis.

Example 1. Assume that we want to find a basis of $\mathbb{R}^4$ relative to the linear span of the vectors $u_1 = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ and $u_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix}$. The reduced column echelon form of the matrix whose columns are these vectors is $\begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & 0 \\ 0 & 1 \\ -\frac{1}{2} & -1 \end{pmatrix}$, so the missing pivots correspond to the second and the fourth row, and the vectors $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ form a relative basis.

Example 2. Furthermore, assume that we want to find a basis of the linear span of the vectors $e_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$, $e_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$ relative to the linear span of the vectors $f_1 = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ and $f_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ (note that $\text{span}(f_1, f_2)$ is a subspace of $\text{span}(e_1, e_2, e_3)$ because $f_1 = -e_1 - e_2$, $f_2 = -e_1 - e_3$). The reduced column echelon form of the matrix whose columns are f_1 and f_2 is the matrix $\begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & 0 \\ 0 & 1 \\ -\frac{1}{2} & -1 \end{pmatrix}$ we computed in the previous example. Now we adjoin the columns equal to e_1, e_2, and e_3, obtaining the matrix $\begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ -\frac{1}{2} & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ -\frac{1}{2} & -1 & 0 & 0 & -1 \end{pmatrix}$. Reducing its three last columns using the first two columns gives the matrix $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 \\ -\frac{1}{2} & -1 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$. Removing the part corresponding to the $\operatorname{span}(u_1, u_2)$ leaves us with the matrix $\begin{pmatrix} 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$ whose reduced column echelon form is $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, so the vector $\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$ forms a relative basis.

Example for $\varphi^2 = 0$

Example 3. Consider the case $V = \mathbb{R}^3$, φ is multiplication by the matrix $A = \begin{pmatrix} -3 & 1 & -1 \\ -12 & 4 & -4 \\ -3 & 1 & -1 \end{pmatrix}$. We have $A^2 = 0$, so $\varphi^2 = 0$, falling into the case we discussed in previous lecture.

We have a sequence of subspaces $V = \operatorname{Ker} \varphi^2 \supset \operatorname{Ker} \varphi \supset \{0\}$. The first one relative to the second one is one-dimensional (since $\dim \operatorname{Ker} \varphi^2 - \dim \operatorname{Ker} \varphi = 1$). The kernel of φ has a basis consisting of the vectors $\begin{pmatrix} 1/3 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1/3 \\ 0 \\ 1 \end{pmatrix}$ (corresponding to the choices $s = 1, t = 0$ and $s = 0, t = 1$ respectively). The reduced column echelon form of the corresponding matrix $A = \begin{pmatrix} 1/3 & -1/3 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the matrix $R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -3 & 1 \end{pmatrix}$, so the missing pivot is in the third row, and we obtain a relative basis consisting of the vector $f = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. This vector gives rise to vector $\varphi(f) = \begin{pmatrix} -1 \\ -4 \\ -1 \end{pmatrix}$. It remains to find a basis of $\operatorname{Ker} \varphi$ relative to the span of $\varphi(f)$. Column reduction of the basis vectors of $\operatorname{Ker}(\varphi)$ by $\varphi(f)$ leaves us with the vector $g = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. Overall, $f, \varphi(f), g$ form a basis of V. The matrix of φ relative to this basis is $A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.