Let us discuss solutions to selected homework questions.

1. (HW1) For each of the following matrices $A$, viewed as a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$,
   - compute the rank of $A$;
   - describe all eigenvalues and eigenvectors of $A$;
   - determine whether there exists a change of coordinates making the matrix $A$ diagonal.

   (a) $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$; (b) $A = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 4 \\ 1 & -1 & 4 \end{pmatrix}$; (c) $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 5 & 2 \end{pmatrix}$.

   **Solution:** (a) $\text{rk}(A) = 1$ (all columns are the same, so there is just one linearly independent column), eigenvectors are 0 and 3, there are two linearly independent eigenvectors \( \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \) for the first of them (and every eigenvector is their linear combination), and every eigenvector for the second of them is proportional to \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \). Since there are three linearly independent eigenvectors, the matrix is similar to a diagonal matrix.

   (b) $\text{rk}(A) = 3$, eigenvectors are 2 and 3, every eigenvector for the first one is proportional to \( \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix} \), every eigenvector for the second one is proportional to \( \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \). Since we do not have three linearly independent eigenvectors, the matrix is not similar to a diagonal matrix.

   (c) $\text{rk}(A) = 3$, eigenvectors are $-2$, 1, and 3, every eigenvector for the first one is proportional to \( \begin{pmatrix} 1/4 \\ 1/2 \\ 1 \end{pmatrix} \), every eigenvector for the second one is proportional to \( \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \), every eigenvector for the third one is proportional to \( \begin{pmatrix} 1/9 \\ 1/3 \\ 1 \end{pmatrix} \).

2. (HW4) For the subspace $U \in \mathbb{R}^5$ spanned by the vectors \( \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \), determine some basis for the subspace $U^\perp$. The scalar product on $\mathbb{R}^5$ is the standard one $(v, w) = v_1w_1 + \ldots + v_5w_5$. 
Solution: The orthogonal complement of our subspace consists of all vectors which are orthogonal to both of the spanning vectors, that is vectors 
\[
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{pmatrix}
\] for which 
\[
x_1 + x_2 + x_3 + x_4 = 0 \quad \text{and} \quad x_1 - x_3 + x_5 = 0.
\]
Solving this system, we observe that the variables \(x_3, x_4, x_5\) are free, and we get a parametrisation of the orthogonal complement:
\[
\begin{pmatrix}
u - w \\
w - 2u - v \\
u \\
v \\
w
\end{pmatrix}, \quad \text{where} \quad u, v, w \in \mathbb{R}.
\]

3. (HW5) Use the Sylvester’s criterion to find all values of the parameter \(a\) for which the quadratic form 
\[
2x_1^2 + x_2^2 + x_3^2 + 2ax_1x_2 + 2x_1x_3 + (2 - 2a)x_2x_3
\] on \(\mathbb{R}^3\) is positive definite.

Solution: The matrix of the corresponding bilinear form is 
\[
A = \begin{pmatrix}
2 & a & 1 \\
a & 1 & 1 - a \\
1 & 1 - a & 1
\end{pmatrix}.
\]
We have \(\Delta_1 = 2, \Delta_2 = 2 - a^2, \Delta_3 = -5a^2 + 6a - 1\). All these numbers are positive if and only if \(|a| < \sqrt{2}\) and \(1/5 < a < 1\) (since the roots of \(-5a^2 + 6a - 1\) are \(1/5\) and \(1\)). In fact, the second condition implies the first one, so we get the answer \(1/5 < a < 1\).

4. (HW6) Find an orthonormal basis of eigenvectors for the matrix 
\[
A = \begin{pmatrix}
2 & 1 \\
1 & 1 \\
1 & 0 \\
0 & 1
\end{pmatrix}.
\]
For the quadratic form \(q(x) = (Ax, x)\), compute the maximal and the minimal value of \(q(x)\) on the unit sphere \(S = \{x \mid (x, x) = 1\}\).

Solution: Eigenvalues are 0, 1, and 3, so the corresponding eigenvectors are automatically orthogonal. Normalizing them (dividing by lengths), we get an orthonormal basis of eigenvectors 
\[
\frac{1}{\sqrt{3}} \begin{pmatrix}
-1 \\
1 \\
1
\end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix}
0 \\
-1 \\
1
\end{pmatrix}, \quad \frac{1}{\sqrt{6}} \begin{pmatrix}
2 \\
1 \\
1
\end{pmatrix}.
\]
We have 
\[
\max_{(x,x)=1} q(x) = 3, \quad \min_{(x,x)=1} q(x) = 0,
\]
as these values are given by the max/min eigenvalue of our matrix. The minimum is reached on the first vector from the basis of eigenvectors, the maximum — on the third vector.