1212: Linear Algebra II

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Lecture 13

Let $B = (b_{ij})$ be the matrix (relative to some basis $e_1, \ldots, e_n$) of a given symmetric bilinear form b on V. We shall now discuss some methods of computing the signature of b via the matrix elements of B.

Theorem 1. *The signature of B is completely determined by eigenvalues of B: the number n_+ is the number of positive eigenvalues, the number n_- is the number of negative eigenvalues, and the number n_0 is the number of zero eigenvalues.*

Proof. We know that $b(x, y) = x^T By$, which of course is equal to $y^T Bx$, since we work with symmetric bilinear forms. Let us pick an orthonormal basis of eigenvectors of the matrix B (with respect to the usual scalar product $(x, y) = y^T x$) $v_1, \ldots, v_n$. Then $b(v_i, v_j) = v_j^T Bv_i = v_j^T c_i v_i = c_i(v_i, v_j)$, therefore, relative to that basis, the matrix of B is diagonal with eigenvalues on the diagonal, and the theorem follows after we normalise each basis vector: $v_i' = \frac{1}{\sqrt{|q(v_i)|}} v_i$. □

Let us denote by B_k the $k \times k$-matrix whose entries are b_{ij} with $1 \leqslant i, j \leqslant k$, that is the top left corner submatrix of B. We put $\Delta_k := \det(B_k)$ for $1 \leqslant k \leqslant n$.

Theorem 2 (Jacobi theorem). *Suppose that for all $i = 1, \ldots, n$ we have $\Delta_i \neq 0$. Then there exists a basis $f_1, \ldots, f_n$ where*

$$q(x_1 f_1 + \cdots + x_n f_n) = \frac{1}{\Delta_1} x_1^2 + \frac{\Delta_1}{\Delta_2} x_2^2 + \cdots + \frac{\Delta_{n-1}}{\Delta_n} x_n^2.$$

Proof. We shall look for a basis of the form

$$\begin{aligned} f_1 &= a_{11} e_1, \\ f_2 &= a_{12} e_1 + a_{22} e_2, \\ &\ldots, \\ f_n &= a_{1n} e_1 + a_{2n} e_2 + \cdots + a_{nn} e_n. \end{aligned}$$

If we write the conditions $b(f_i, f_j) = 0$ for $i \neq j$ directly, we shall obtain a system of quadratic equations in the unknowns a_{ij}, which is difficult to solve directly. For that reason, we shall use a clever shortcut.

Suppose that we found a basis of the form given above, for which

$$b(f_i, e_j) = 0 \text{ for } j = 1, \ldots, i-1.$$

We shall now verify that these conditions imply $b(f_i, f_j) = 0$ for $i \neq j$. Indeed, for $i > j$ we have

$$b(f_i, f_j) = b(f_i, a_{1j} e_1 + a_{2j} e_2 + \ldots + a_{jj} e_j) = a_{1j} b(f_i, e_1) + \cdots + a_{jj} b(f_i, e_j) = 0,$$

and for $i < j$ we have $b(f_i, f_j) = b(f_j, f_i) = 0$.

For a given i, the conditions

$$b(f_i, e_j) = 0 \text{ for } j = 1, \ldots, i-1$$

form a system of linear equations with i unknowns and $i-1$ equations, so there will inevitably be free unknowns. To normalise the solution, let us also include the equation

$$b(f_i, e_i) = 1.$$

Then the corresponding system of equation becomes

$$\begin{cases} b(e_1, e_1)a_{1i} + b(e_2, e_1)a_{2i} + \ldots + b(e_i, e_1)a_{ii} = 0, \\ b(e_1, e_2)a_{1i} + b(e_2, e_2)a_{2i} + \ldots + b(e_i, e_2)a_{ii} = 0, \\ \ldots \\ b(e_1, e_{i-1})a_{1i} + b(e_2, e_{i-1})a_{2i} + \ldots + b(e_i, e_{i-1})a_{ii} = 0, \\ b(e_1, e_i)a_{1i} + b(e_2, e_i)a_{2i} + \ldots + b(e_i, e_i)a_{ii} = 1. \end{cases}$$

The matrix of the this system of equation is $B_i^T = B_i$, so by our assumption this system has just one solution for each $i = 1, \ldots, n$.

Let us compute the diagonal entries $b(f_i, f_i)$. We have

$$b(f_i, f_j) = b(f_i, a_{1j}e_1 + a_{2j}e_2 + \ldots + a_{ii}e_i) = a_{1j}b(f_i, e_1) + \cdots + a_{ii}b(f_i, e_i) = a_{ii}.$$

To compute a_{ii}, we use the Cramer's rule for solving systems of linear equations. The last unknown is equal to the ratio $\frac{\det(B_{ii})}{\det(B_i)}$, where B_{ii} is obtained by B_i by replacing the last column by the right hand side of the given system of equations. Expanding that determinant along the right column, we get $a_{ii} = \frac{\Delta_{i-1}}{\Delta_i}$ for $i > 1$, and $a_{11} = \frac{1}{\Delta_1}$, as required. □