Bilinear and quadratic forms

Recall the definition from the last class.

**Definition 1.** Let $V$ be a vector space. A function $V \times V \to \mathbb{R}$, $v_1, v_2 \mapsto b(v_1, v_2)$ is called a bilinear form if for all vectors $v, v_1, v_2$ the following conditions are satisfied:

$$b(c_1 v_1 + c_2 v_2, v) = c_1 b(v_1, v) + c_2 b(v_2, v)$$

and

$$b(v, c_1 v_1 + c_2 v_2) = c_1 b(v, v_1) + c_2 b(v, v_2).$$

A bilinear form is said to be **symmetric** if $b(v_1, v_2) = b(v_2, v_1)$ for all $v_1, v_2$, and **skew-symmetric** if $b(v_1, v_2) = -b(v_2, v_1)$ for all $v_1, v_2$. A symmetric bilinear form is said to be **positive semidefinite** if $b(v, v) \geq 0$ for all $v$, and **positive definite**, if $b(v, v) > 0$ for $v \neq 0$. In these words, a function of two vector arguments is a scalar product if and only if it is bilinear, symmetric, and positive definite.

**Remark 1.** Generalising what we proved about scalar products, for every bilinear form $b$ and every basis $e_1, \ldots, e_n$ of $V$, we have

$$b(x_1 e_1 + \ldots + x_n e_n, y_1 e_1 + \ldots + y_n e_n) = \sum_{i,j=1}^{n} b_{ij} x_i y_j,$$

where $b_{ij} = b(e_i, e_j)$. Moreover, this number corresponds to the $1 \times 1$-matrix $x^T B y$, where $B$ is the matrix with entries $b_{ij}$.

Every bilinear form $b$ gives rise to a quadratic form by putting $q(x) = b(x, x)$, for example, the bilinear form

$$b(x_1 e_1 + x_2 e_2, y_1 e_1 + y_2 e_2) = 2x_1 y_2$$

gives rise to a quadratic form $2x_1 x_2$, and the bilinear form

$$b(x_1 e_1 + x_2 e_2, y_1 e_1 + y_2 e_2) = x_1 y_2 + x_2 y_1$$

gives rise to the same quadratic form. It turns out that the reconstruction of $b$ from $q$ is unique if we assume that $b$ is symmetric; in this case the reconstruction formula is

$$b(v, w) := \frac{1}{2} (q(v + w) - q(v) - q(w)).$$

Indeed, if $q(v) = b(v, v)$, then

$$\frac{1}{2} (q(v + w) - q(v) - q(w)) = \frac{1}{2} (b(v + w, v + w) - b(v, v) - b(w, w)) =$$

$$= \frac{1}{2} (b(v, v) + b(v, w) + b(w, v) + b(w, w) - b(v, v) - b(w, w)) = \frac{1}{2} (b(v, w) + b(w, v)),$$

which, for a symmetric bilinear form, is $b(v, w)$. 

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The coefficients $a_{ij}$ of a quadratic form and the coefficients $b_{ij}$ of the corresponding symmetric bilinear form are related by $a_{ii} = b_{ii}$ and $a_{ij} = b_{ij} + b_{ji} = 2b_{ij}$ for $i < j$.

We shall now formulate several theorems about quadratic forms and symmetric bilinear forms; they will be a topic of our tutorial class and the next problem sheet, and next week we shall discuss their proofs in detail.

One celebrated example of a quadratic form is $q(x_1, x_2, x_3, t) = x_1^2 + x_2^2 + x_3^2 - t^2$ on the Minkowski space $\mathbb{R}^4$, it is used in special relativity theory. This serves as a (humble) motivation for the following result.

**Theorem 1.** Let $q$ be a quadratic form on a vector space $V$. There exists a basis $f_1, \ldots, f_n$ of $V$ for which the quadratic form $q$ becomes a signed sum of squares:

$$q(x_1 f_1 + \cdots + x_n f_n) = \sum_{i=1}^{n} \varepsilon_i x_i^2,$$

where all numbers $\varepsilon_i$ are either 1 or −1 or 0.

**Theorem 2** (Law of inertia). In the previous theorem, the triple $(n_+, n_-, n_0)$, where $n_\pm$ is the number of $\varepsilon_i$ equal to $\pm 1$, and $n_0$ is the number of $\varepsilon_i$ equal to 0, does not depend on the choice of the basis $f_1, \ldots, f_n$. This triple is often referred to as the signature of the quadratic form $q$.

Let $B = (b_{ij})$ be the matrix of a given symmetric bilinear form $b$ on $V$. We shall now discuss some methods of computing the signature of $b$ via the matrix elements of $B$.

**Theorem 3.** The signature of $B$ is completely determined by eigenvalues of $B$: the number $n_+$ is the number of positive eigenvalues, the number $n_-$ is the number of negative eigenvalues, and the number $n_0$ is the number of zero eigenvalues.

Note that this theorem makes sense because all eigenvalues of a symmetric matrix are real.

Let us denote by $B_k$ the $k \times k$-matrix whose entries are $b_{ij}$ with $1 \leq i, j \leq k$, that is the top left corner submatrix of $B$. We put $\Delta_0 = 1$ and $\Delta_k := \det(B_k)$ for $1 \leq k \leq n$.

**Theorem 4** (Jacobi diagonal form). Suppose that for all $i = 1, \ldots, n$ we have $\Delta_i \neq 0$. Then there exists a system of coordinates where the matrix of $b$ is a diagonal matrix with the numbers $\frac{\Delta_{i+1}}{\Delta_i}$ on the diagonal.

**Theorem 5** (Sylvester's criterion). The given symmetric bilinear form is positive definite if and only if

$$\Delta_k > 0 \quad \text{for all} \quad k = 1, \ldots, n.$$