MA 1111/1212: Linear Algebra
Tutorial problems, December 4, 2014

**1.** (**a**) Let us compute the images of the basis vectors: $1 \mapsto (3-7i)\cdot 1 = 3-7i)$, $i \mapsto (3-7i)\cdot i = 7+3i$. This instantly leads to the matrix $\begin{pmatrix} 3 & 7 \\ -7 & 3 \end{pmatrix}$.

(**b**) Let us compute the images of the basis vectors:

$$\begin{aligned}
e_1 &\mapsto \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} = -e_1 - e_2 + e_3,\\
e_2 &\mapsto \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -e_1 + e_4,\\
e_3 &\mapsto \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = e_1 - e_4,\\
e_4 &\mapsto \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} = e_2 - e_3 + e_4.
\end{aligned}$$

This immediately leads to the matrix

$$\begin{pmatrix} -1 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 1 \end{pmatrix}.$$

**2.** (**a**) Let us compute the transition matrix $M_{\mathbf{e},\mathbf{f}}$. Solving the systems of equations $f_1 = c_{11}e_1 + c_{21}e_2$ and $f_2 = c_{12}e_1 + c_{22}e_2$, we get $c_{11} = 7$, $c_{21} = -3$, $c_{12} = 30$, $c_{22} = -13$, therefore $M_{\mathbf{e},\mathbf{f}} = \begin{pmatrix} 7 & 30 \\ -3 & -13 \end{pmatrix}$. Therefore, we have

$$A_{\varphi,\mathbf{f}} = M_{\mathbf{e},\mathbf{f}}^{-1} A_{\varphi,\mathbf{e}} M_{\mathbf{e},\mathbf{f}} = \begin{pmatrix} 171 & 731 \\ -40 & -171 \end{pmatrix}.$$

(**b**) We clearly have $M_{\mathbf{v},\mathbf{e}} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$, therefore $M_{\mathbf{e},\mathbf{v}} = M_{\mathbf{v},\mathbf{e}}^{-1} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$. Therefore,

$$A_{\varphi,\mathbf{v}} = M_{\mathbf{e},\mathbf{v}}^{-1} A_{\varphi,\mathbf{e}} M_{\mathbf{e},\mathbf{v}} = \begin{pmatrix} 5 & -3 \\ 8 & -5 \end{pmatrix}.$$

**3.** As the hint suggests, $Ax = c \cdot x$ can be rewritten as $(A - c \cdot I_2)x = 0$. This system has a nontrivial solution if and only if $\det(A - c\cdot I_2) = 0$. Writing $A = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$, we rewrite $\det(A - c \cdot I_2)$ as

$$\det\begin{pmatrix} x-c & y \\ z & t-c \end{pmatrix} = (x-c)(t-c)-yz = (xt-yz)-c(x+t)+c^2 = c^2 = c\,\mathrm{tr}(A)+\det(A),$$

as required.