MA 1111/1212: Linear Algebra
Tutorial problems, October 16, 2014

1. Let us write down the corresponding one-row representatives of these permutations (for which we permute the columns to create the natural order in the top row): the matrix $\begin{pmatrix}1&2&3&4&5\\2&1&5&4&3\end{pmatrix}$ corresponds to $2,1,5,4,3$, the matrix $\begin{pmatrix}1&4&2&3&5\\2&1&5&3&4\end{pmatrix}$ corresponds to $2,5,3,1,4$, and the matrix $\begin{pmatrix}5&3&1&4&2\\3&5&2&4&1\end{pmatrix}$ corresponds to $2,1,5,4,3$. Therefore the first and the third matrix do represent the same permutation. The permutation $2,1,5,4,3$ is even (it has 4 inversions), and the permutation $2,5,3,1,4$ is odd (it has 3 inversions).

2. Clearly, we must have $i = 2$ (to have all the numbers present in the top row) and $\{j,k\} = \{3,4\}$. For the choice $j = 3$, $k = 4$, the permutation is even (since there are 8 inversions in total in the two rows), so for the other choice the permutation is odd. *Answer:* $i = 2$, $j = 4$, $k = 3$.

3. The last two rows are proportional, so subtracting from the third row three times the second row, we obtain a matrix with a zero row whose determinant is zero.

4. (**a**) If we subtract from the second row twice the first row, and add to the third row the first row, we get the matrix $\begin{pmatrix}1&1&1\\0&-1&-2\\0&6&3\end{pmatrix}$; now, we have

$$\det\begin{pmatrix}1&1&1\\0&-1&-2\\0&6&3\end{pmatrix} = -3\det\begin{pmatrix}1&1&1\\0&1&2\\0&2&1\end{pmatrix} = -3\det\begin{pmatrix}1&0&-1\\0&1&2\\0&0&-3\end{pmatrix} = 9\det\begin{pmatrix}1&0&-1\\0&1&2\\0&0&1\end{pmatrix} = 9.$$

(**b**) We have

$$\det\begin{pmatrix}2&1&-3&0\\1&5&2&-1\\5&0&13&8\\0&1&2&1\end{pmatrix} = -\det\begin{pmatrix}1&5&2&-1\\2&1&-3&0\\5&0&13&8\\0&1&2&1\end{pmatrix} = -\det\begin{pmatrix}1&5&2&-1\\0&-9&-7&2\\0&-25&3&13\\0&1&2&1\end{pmatrix} =$$

$$\det\begin{pmatrix}1&5&2&-1\\0&1&2&1\\0&-25&3&13\\0&-9&-7&2\end{pmatrix} = \det\begin{pmatrix}1&5&2&-1\\0&1&2&1\\0&0&53&38\\0&0&11&11\end{pmatrix} = 11\det\begin{pmatrix}1&5&2&-1\\0&1&2&1\\0&0&53&38\\0&0&1&1\end{pmatrix} =$$

$$-11\det\begin{pmatrix}1&5&2&-1\\0&1&2&1\\0&0&1&1\\0&0&53&38\end{pmatrix} = -11\det\begin{pmatrix}1&5&2&-1\\0&1&2&1\\0&0&1&1\\0&0&0&-15\end{pmatrix} = 11\cdot 15\det\begin{pmatrix}1&5&2&-1\\0&1&2&1\\0&0&1&1\\0&0&0&1\end{pmatrix} = 165.$$

Optional question: Subtracting, for each $i = 1,\dots,n-1$, the row i from the row $i+1$, we obtain the matrix whose diagonal entry are all equal to 1, and all entries below diagonal are equal to zero; its determinant is equal to 1.