MA 1111: Linear Algebra I
Tutorial problems, October 2, 2014

1. The matrix in question clearly is

$$\begin{pmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \end{pmatrix}.$$

Let us compute the reduced row echelon form of this matrix. We first subtract one-third of the first row from the two others, and then multiply that row by one-third:

$$\begin{pmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \end{pmatrix} \overset{(2)-1/3(1),(3)-1/3(1),1/3(1)}{\longmapsto} \begin{pmatrix} 1 & 1/3 & 1/3 & 1/3 \\ 0 & 8/3 & 2/3 & 2/3 \\ 0 & 2/3 & 8/3 & 2/3 \end{pmatrix}.$$

We now subtract one-fourth of the second row from the third one, and then multiply the second row by $3/8$:

$$\begin{pmatrix} 1 & 1/3 & 1/3 & 1/3 \\ 0 & 8/3 & 2/3 & 2/3 \\ 0 & 2/3 & 8/3 & 2/3 \end{pmatrix} \overset{(3)-1/4(2),3/8(2)}{\longmapsto} \begin{pmatrix} 1 & 1/3 & 1/3 & 1/3 \\ 0 & 1 & 1/4 & 1/4 \\ 0 & 0 & 5/2 & 1/2 \end{pmatrix}.$$

Now, we multiply the third row by $2/5$. The result is a matrix in row echelon form but not reduced row echelon form:

$$\begin{pmatrix} 1 & 1/3 & 1/3 & 1/3 \\ 0 & 1 & 1/4 & 1/4 \\ 0 & 0 & 1 & 1/5 \end{pmatrix}.$$

To bring it to reduced row echelon form, we first subtract from the second row one-fourth of the third row, and from the first row one-third of the third row:

$$\begin{pmatrix} 1 & 1/3 & 1/3 & 1/3 \\ 0 & 1 & 1/4 & 1/4 \\ 0 & 0 & 1 & 1/5 \end{pmatrix} \overset{(2)-1/4(3),(1)-1/3(3)}{\longmapsto} \begin{pmatrix} 1 & 1/3 & 0 & 4/15 \\ 0 & 1 & 0 & 1/5 \\ 0 & 0 & 1 & 1/5 \end{pmatrix}$$

Finally, we subtract from the first row one-third of the second row:

$$\begin{pmatrix} 1 & 1/3 & 0 & 4/15 \\ 0 & 1 & 0 & 1/5 \\ 0 & 0 & 1 & 1/5 \end{pmatrix} \overset{(1)-1/3(2)}{\longmapsto} \begin{pmatrix} 1 & 0 & 0 & 1/5 \\ 0 & 1 & 0 & 1/5 \\ 0 & 0 & 1 & 1/5 \end{pmatrix}.$$

We conclude that the reduced row echelon form of this matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 1/5 \\ 0 & 1 & 0 & 1/5 \\ 0 & 0 & 1 & 1/5 \end{pmatrix},$$

so the only solution is $x = y = z = 1/5$.

2. (**a**) Explain why the matrix

$$A = \begin{pmatrix} 1 & 1 & 2 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

is not a in reduced row echelon form.

The pivot of the second row has a non-zero element above, and also the pivot of the last row has non-zero elements above.

To bring A to its reduced row echelon form, we subtract twice the second row from the first row, the last row from the first row, and twice the last row from the third row:

$$\begin{pmatrix} 1 & 1 & 2 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \overset{(1)-2(2),(1)-(4),(3)-2(4)}{\longmapsto} \begin{pmatrix} 1 & 1 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

The result is the reduced row echelon form. The system of linear equations corresponding to A is

$$\begin{cases} x_1 + x_2 + 2x_3 + x_6 = 1, \\ x_3 + x_4 = 0, \\ x_5 + 2x_6 = 3, \\ x_6 = 1. \end{cases}$$

The equivalent system of equations is

$$\begin{cases} x_1 + x_2 - 2x_4 = 0, \\ x_3 + x_4 = 0, \\ x_5 = 1, \\ x_6 = 1. \end{cases}$$

The free unknowns are x_2 and x_4, the others are pivotal. Assigning arbitrary parameters to x_2 and x_4, we get the solution set

$$\begin{aligned} x_1 &= -t_2 + 2t_4, \\ x_2 &= t_2, \\ x_3 &= -t_4, \\ x_4 &= t_4, \\ x_5 &= 1, \\ x_6 &= 1. \end{aligned}$$

3. Let us solve this system of equations by the usual methods. The corresponding matrix is

$$\begin{pmatrix} 2 & -1 & 4 & 1 \\ 7 & 2 & 1 & 5 \end{pmatrix},$$

and it is brought to its reduced row echelon form as follows:

$$\begin{pmatrix} 2 & -1 & 4 & 1 \\ 7 & 2 & 1 & 5 \end{pmatrix} \overset{(2)-7/2(1),1/2(1)}{\longmapsto} \begin{pmatrix} 1 & -1/2 & 2 & 1/2 \\ 0 & 11/2 & -13 & 3/2 \end{pmatrix} \overset{(1)+1/11(2),2/11(2)}{\longmapsto} \begin{pmatrix} 1 & 0 & 9/11 & 7/11 \\ 0 & 1 & -26/11 & 3/11 \end{pmatrix}.$$

This shows that z is the free unknown of this system. Assigning an arbitrary value of t to it, we get a parametric representation of the solution set

$$\begin{cases} x = 7/11 - 9/11t, \\ y = 3/11 + 26/11t, \\ z = t. \end{cases}$$