MA 1111/1212: Linear Algebra
Tutorial problems, December 4, 2014

1. (**a**) The set of all complex numbers forms a 2-dimensional (real) vector space with a basis $1, i$. Compute the matrix (relative to this basis) of linear operator on this space that maps every complex number z to $(3-7i)z$.

(**b**) The space of all 2×2-matrices forms a 4-dimensional vector space with a basis $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Compute the matrix (relative to this basis) of linear operator on this space that maps every 2×2-matrix X to $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \cdot X - X \cdot \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$.

2. Let $V = \mathbb{R}^2$, $e_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $e_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ a basis of V, $\varphi \colon V \to V$ a linear transformation whose matrix $A_{\varphi,\mathbf{e}}$ relative to the basis e_1, e_2 is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(**a**) Find the transition matrix $M_{\mathbf{ef}}$ from the basis e_1, e_2 to the basis $f_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, $f_2 = \begin{pmatrix} 4 \\ -9 \end{pmatrix}$, and compute the matrix $A_{\varphi,\mathbf{f}}$.

(**b**) Compute the matrix $A_{\varphi,\mathbf{v}}$ of the linear transformation φ relative to the basis of standard unit vectors $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

3. Let c be a scalar. Show that for a 2×2-matrix A, the system of equations $Ax = c \cdot x$ has a nontrivial solution $x \neq 0$ if and only if

$$c^2 - \operatorname{tr}(A)c + \det(A) = 0.$$

(*Hint:* write that system of equations as $(A - c \cdot I_2)x = 0$).