1111: Linear Algebra I

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Lecture 22

Computing Fibonacci numbers

Fibonacci numbers are defined recursively: $f_0 = 0$, $f_1 = 1$, $f_n = f_{n-1} + f_{n-2}$ for $n \geqslant 2$, so that this sequence starts like this:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

I shall now explain how to derive a formula for these using linear algebra.

Idea 1: let us consider a much simpler question: let $g_0 = 1$, and $g_n = cg_{n-1}$ for $n \geqslant 1$. Then of course $g_n = c^n$.

In our case, each of the numbers is determined by two previous ones, let us store pairs! We put $v_n = \begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix}$.

Then

$$v_{n+1} = \begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix} = \begin{pmatrix} f_n \\ f_n + f_{n-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} f_{n-1} \\ f_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} v_n,$$

therefore

$$v_{n+1} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} v_n = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} v_{n-1} = \cdots = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n v_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{n+1} v_0 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{n+1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Therefore, we shall be able to compute Fibonacci numbers if we can compute the n-th power of the matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.

Idea 2: If our matrix were a diagonal matrix $\begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}$, its n-th power would have been $\begin{pmatrix} b_1^n & 0 \\ 0 & b_2^n \end{pmatrix}$. But our matrix is not like that. What shall we do? That's where linear algebra is particularly beneficial: we shall view this matrix as a matrix of a linear transformation, and change coordinates: find another system of coordinates where the matrix representing this transformation is $\begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}$ for some b_1 and b_2.

What does it mean for a matrix of a linear operator φ to be diagonal in the system of coordinates given by the basis e_1, e_2? This means $\varphi(e_1) = b_1 e_1$, $\varphi(e_2) = b_2 e_2$.

Definition 1. For a linear transformation $\varphi\colon V \to V$, a nonzero vector v satisfying $\varphi(v) = c \cdot v$ for some scalar c is called an *eigenvector* of φ. The number c is called an *eigenvalue* of φ.

Lemma 1. *Let φ be a linear transformation, and let A be the matrix of φ relative to some basis $e_1, \ldots, e_n$. A number c is an eigenvalue of φ if and only if* $\det(A - cI_n) = 0$.

Proof. Suppose that c is an eigenvalue, which happens if and only if there exists a nonzero vector v such that $\varphi(v) = c \cdot v$. In coordinates relative to the appropriate basis, $A \cdot v_e = c \cdot v_e$, or, in other words, $(A - cI_n) \cdot v_e = 0$. Therefore, c is an eigenvalue if and only if the system of equations $(A - cI_n) \cdot x = 0$ has a nontrivial solution, which happens if and only if the matrix $A - cI_n$ is not invertible, which happens if and only if $\det(A - cI_n) = 0$. □

As you saw in the tutorial sheet a couple of hours ago, in the case of 2×2-matrices, this means that the eigenvalues are roots of the equation $t^2-\mathrm{tr}(A)t+\det(A)=0$. In our case, this means that $t^2-t-1=0$, so the eigenvalues are $\frac{1\pm\sqrt{5}}{2}$.

The corresponding eigenvectors are obtained from solutions of the systems of equations $Ax=\frac{1\pm\sqrt{5}}{2}x$. The first of them has the general solution $\begin{pmatrix} x_1 \\ \frac{1+\sqrt{5}}{2}x_1\end{pmatrix}$, and the second one has the general solution $\begin{pmatrix} x_1 \\ \frac{1-\sqrt{5}}{2}x_1\end{pmatrix}$. Setting in each cases $x_1=1$, we obtain two eigenvectors $\mathbf{e}_1=\begin{pmatrix} 1 \\ \frac{1+\sqrt{5}}{2}\end{pmatrix}$ and $\mathbf{e}_2=\begin{pmatrix} 1 \\ \frac{1-\sqrt{5}}{2}\end{pmatrix}$. The transition matrix from the basis of standard unit vectors $\mathbf{s}_1$, $\mathbf{s}_2$ to this basis is, manifestly, $M_{\mathbf{s},\mathbf{e}}=\begin{pmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2}\end{pmatrix}$, so

$$M_{\mathbf{s},\mathbf{e}}^{-1}=-\frac{1}{\sqrt{5}}\begin{pmatrix} \frac{1-\sqrt{5}}{2} & -1 \\ \frac{-1-\sqrt{5}}{2} & 1\end{pmatrix},$$

Since $A\mathbf{e}_1=(\frac{1+\sqrt{5}}{2})\mathbf{e}_1$, and $A\mathbf{e}_2=(\frac{1-\sqrt{5}}{2})\mathbf{e}_2$, the matrix of the linear transformation φ relative to the basis $\mathbf{e}_1$, $\mathbf{e}_2$ is

$$M_{\mathbf{s},\mathbf{e}}^{-1}AM_{\mathbf{s},\mathbf{e}}=\begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2}\end{pmatrix}.$$

Therefore,

$$A=M_{\mathbf{s},\mathbf{e}}\begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2}\end{pmatrix}M_{\mathbf{s},\mathbf{e}}^{-1},$$

and hence

$$A^n=\left(M_{\mathbf{s},\mathbf{e}}\begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2}\end{pmatrix}M_{\mathbf{s},\mathbf{e}}^{-1}\right)^n=M_{\mathbf{s},\mathbf{e}}\begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2}\end{pmatrix}^n M_{\mathbf{s},\mathbf{e}}^{-1}=M_{\mathbf{s},\mathbf{e}}\begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n\end{pmatrix}M_{\mathbf{s},\mathbf{e}}^{-1}.$$

Substituting the above formulas for $M_{\mathbf{s},\mathbf{e}}$ and $M_{\mathbf{s},\mathbf{e}}^{-1}$, we see that

$$A^n=\begin{pmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2}\end{pmatrix}\begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n\end{pmatrix}-\frac{1}{\sqrt{5}}\begin{pmatrix} \frac{1-\sqrt{5}}{2} & -1 \\ \frac{-1-\sqrt{5}}{2} & 1\end{pmatrix}$$

In fact, we have $\mathbf{v}_n=A^n\mathbf{v}_0$, so

$$\mathbf{v}_n=\begin{pmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2}\end{pmatrix}\begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n\end{pmatrix}\left(-\frac{1}{\sqrt{5}}\right)\begin{pmatrix} \frac{1-\sqrt{5}}{2} & -1 \\ \frac{-1-\sqrt{5}}{2} & 1\end{pmatrix}\begin{pmatrix} 0 \\ 1\end{pmatrix}=$$

$$=\begin{pmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2}\end{pmatrix}\begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n\end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}}\end{pmatrix}=\begin{pmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2}\end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n \\ -\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n\end{pmatrix}=$$

$$=\begin{pmatrix} \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right) \\ \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right)\end{pmatrix}$$

Recalling that $\mathbf{v}_n=\begin{pmatrix} f_n \\ f_{n+1}\end{pmatrix}$, we observe that

$$f_n=\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$