Examples of linear maps and coordinate changes

Example 1. Let \( P_n \) be the vector space of all polynomials in one variable \( x \) of degree at most \( n \). Then there is a function \( \varphi: P_n \to P_{n+1} \) that maps every polynomial \( f(x) \) to \( xf(x) \). (Note that the target of \( \varphi \) has to be different, since multiplying by \( x \) increases degrees). This function is a linear map, which we can check in the same way as we did in previous class.

Example 2. Let \( P_n \) be the vector space of all polynomials in one variable \( x \) of degree at most \( n \). Then we can define both a function \( \psi: P_n \to P_n \) that maps every polynomial \( f(x) \) to \( f'(x) \), and a function \( \hat{\psi}: P_n \to P_n \) that every polynomial \( f(x) \) to \( f'(x) \) (since the degree of the derivative of a polynomial of degree at most \( n \) is at most \( n-1 \)). These functions are linear maps, which we can check in the same way as in previous class. In fact, \( \hat{\psi} \) is a linear transformation, since it is a map from \( P_n \) to itself.

Example 3. Consider the vector space \( M_2 \) of all \( 2 \times 2 \)-matrices. Let us define a function \( \alpha: M_2 \to M_2 \) by the formula \( \alpha(X) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} X \). Let us check that this map is a linear transformation. Indeed, by properties of matrix products

\[
\alpha(X_1 + X_2) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} (X_1 + X_2) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} X_1 + \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} X_2 = \alpha(X_1) + \alpha(X_2),
\]

\[
\alpha(cX) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} (cX) = c \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} X = c\alpha(X).
\]

Example 4. Let us consider the linear map \( \varphi \) from Example 1, and assume \( n = 2 \). Let us take the bases \( e_1 = 1, e_2 = x, e_3 = x^2 \) of \( P_2 \), and let us take the bases \( f_1 = 1, f_2 = x, f_3 = x^2 \) of \( P_3 \), and compute \( A_{\varphi,e,f} \). Note that \( \varphi(e_1) = 1 \cdot 1 = x = f_2, \varphi(e_2) = x \cdot x = x^2 = f_3, \) and \( \varphi(e_3) = x \cdot x^2 = x^3 = f_4 \). Therefore

\[
A_{\varphi,e,f} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.
\]

Example 5. Let us consider the linear maps \( \psi \) and \( \hat{\psi} \) from Example 2, and assume \( n = 3 \). Let us take the bases \( e_1 = 1, e_2 = x, e_3 = x^2, e_4 = x^3 \) of \( P_3 \), and the basis \( f_1 = 1, f_2 = x, f_3 = x^2 \) of \( P_2 \), and let us compute \( A_{\psi,e,f} \) and \( A_{\hat{\psi},e,f} \). Note that \( \psi(e_1) = 1' = 0, \psi(e_2) = x' = 1 = f_1, \psi(e_3) = (x^2)' = 2x = 2f_2, \) and \( \psi(e_4) = (x^3)' = 3x^2 = 3f_3 \), and that \( \hat{\psi}(e_1) = 1' = 0, \hat{\psi}(e_2) = x' = 1 = e_1, \hat{\psi}(e_3) = (x^2)' = 2x = 2e_2, \) and \( \hat{\psi}(e_4) = (x^3)' = 3x^2 = 3e_3 \). Therefore

\[
A_{\psi,e,f} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.
\]
and
\[
A_{\psi, e} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 3 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

**Example 6.** Let us look at the linear map \( \alpha \) from Example 3. We consider the basis of matrix units in \( M_2 \): \( e_1 = E_{11} \), \( e_2 = E_{12} \), \( e_3 = E_{21} \), \( e_4 = E_{22} \). We have
\[
\alpha(e_1) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = e_1 + e_3,
\]
\[
\alpha(e_2) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} e_2 + e_4,
\]
\[
\alpha(e_3) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = e_1,
\]
\[
\alpha(e_4) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = e_2,
\]
so
\[
A_{\alpha, e} = \begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}.
\]

**Example 7.** Let us take two bases of \( \mathbb{R}^2 \): \( e_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \), \( e_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \), and \( f_1 = \begin{pmatrix} 7 \\ 5 \end{pmatrix} \), \( f_2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \). Suppose that the matrix of a linear transformation \( \varphi : \mathbb{R}^2 \to \mathbb{R}^2 \) relative to the first basis is \( \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix} \). Let us compute its matrix relative to the second basis. For that, we first compute the transition matrix \( M_{e,f} \). We have
\[
f_1 = \begin{pmatrix} 7 \\ 5 \end{pmatrix} = 5e_1 + 2e_2,
\]
\[
f_2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix} = 3e_1 + e_2,
\]
so
\[
M_{e,f} = \begin{pmatrix} 5 & 3 \\ 2 & 1 \end{pmatrix},
\]
and
\[
M_{e,f}^{-1} = \begin{pmatrix} -1 & 3 \\ 2 & -5 \end{pmatrix}.
\]
Therefore
\[
A_{\varphi, f} = M_{e,f}^{-1}A_{\varphi, e}M_{e,f} = \begin{pmatrix} -1 & 3 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 5 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} -2 & -3 \\ 10 & 9 \end{pmatrix}.
\]

**Computing Fibonacci numbers**

Fibonacci numbers are defined recursively: \( f_0 = 0 \), \( f_1 = 1 \), \( f_n = f_{n-1} + f_{n-2} \) for \( n \geq 2 \), so that this sequence starts like this:
\[
0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots
\]
Next time we shall discuss how to derive a formula for these using linear algebra.