Linear maps and change of coordinates

As a last step, let us exhibit how matrices of linear maps transform under changes of coordinates.

**Lemma 1.** Let \( \varphi : V \to W \) be a linear operator, and suppose that \( e_1, \ldots, e_n \) and \( e'_1, \ldots, e'_n \) are two bases of \( V \), and \( f_1, \ldots, f_m \) and \( f'_1, \ldots, f'_m \) are two bases of \( W \). Then

\[
A_{\varphi, e', f'} = M_{f', f}A_{\varphi, e, f}M_{e, e'}^{-1}.
\]

**Proof.** Let us take a vector \( v \in V \). On the one hand, the formula of Lemma 2 tells us that \((\varphi(v))_{f'} = A_{\varphi, e', f'}v_{e'}\).

On the other hand, applying various results we proved earlier, we have

\[
(\varphi(v))_{f'} = M_{f', f}(A_{\varphi, e, f}v_e) = M_{f', f}(A_{\varphi, e, f}(M_{e, e'}v_{e'})) = (M_{f', f}A_{\varphi, e, f}M_{e, e'})v_{e'}.
\]

Therefore,

\[
A_{\varphi, e', f'}v_{e'} = (M_{f', f}A_{\varphi, e, f}M_{e, e'})v_{e'}
\]

for every \( v_{e'} \). From our previous classes we know that knowing \( Av \) for all vectors \( v \) completely determines the matrix \( A \), so

\[
A_{\varphi, e', f'} = (M_{f', f}A_{\varphi, e, f}M_{e, e'})^{-1} = (M_{f, f}A_{\varphi, e, f}M_{e, e'})^{-1}
\]

because of properties of transition matrices proved earlier. \( \square \)

**Remark 1.** Our formula

\[
A_{\varphi, e', f'} = M_{f', f}A_{\varphi, e, f}M_{e, e'}
\]

shows that changing from the coordinate systems \( e, f \) to some other coordinate system amounts to multiplying the matrix \( A_{\varphi, e, f} \) by some invertible matrices on the left and on the right, so effectively to performing a certain number of elementary row and column operations on this matrix. This is very useful (but not applicable to a more narrow class of linear transformations, see below).

**Remark 2.** A linear operator \( \varphi : V \to V \) is often called a linear transformation. For a linear transformation, it makes sense to use the same coordinate system for the input and the output. By definition, the matrix of a linear operator \( \varphi : V \to V \) relative to the basis \( e_1, \ldots, e_n \) is

\[
A_{\varphi, e} := A_{\varphi, e, e}.
\]

**Lemma 2.** For a linear transformation \( \varphi : V \to V \), and two bases \( e_1, \ldots, e_n \) and \( e'_1, \ldots, e'_n \) of \( V \), we have

\[
A_{\varphi, e'} = M^{-1}_{e, e'}A_{\varphi, e}M_{e, e'}.
\]

**Proof.** This is a particular case of Lemma 1. \( \square \)
Remark 3. Proposition 5 shows that for a square matrix $A$, the change $A \mapsto C^{-1}AC$ with an invertible matrix $C$, corresponds to the situation where $A$ is viewed as a matrix of a linear transformation, and $C$ is viewed as a transition matrix for a coordinate change. You verified in your earlier home assignments that $\text{tr}(C^{-1}AC) = \text{tr}(A)$ and $\det(C^{-1}AC) = \det(A)$; these properties imply that the trace and the determinant do not depend on the choice of coordinates, and hence reflect some geometric properties of a linear transformation. (In case of the determinant, those properties have been hinted at in our previous classes: determinants compute how a linear transformation changes volumes of solids).

Examples of linear maps and coordinate changes

Example 1. As we know, every linear map $\varphi: \mathbb{R}^n \to \mathbb{R}^k$ is given by a $k \times n$-matrix $A$, so that $\varphi(x) = Ax$.

Example 2. Let $V$ be the vector space of all polynomials in one variable $x$. Consider the function $\varphi: V \to V$ that maps every polynomial $f(x)$ to $xf(x)$. This is a linear map:

$$x(f_1(x) + f_2(x)) = xf_1(x) + xf_2(x),$$
$$x(cf(x)) = c(xf(x)).$$

Example 3. Let $V$ be the vector space of all polynomials in one variable $x$. Consider the function $\psi: V \to V$ that maps every polynomial $f(x)$ to $f'(x)$. This is a linear map:

$$(f_1(x) + f_2(x))' = f_1'(x) + f_2'(x),$$
$$(cf(x))' = cf'(x).$$

Example 4. Let $V$ be the vector space of all polynomials in one variable $x$. Consider the function $\alpha: V \to V$ that maps every polynomial $f(x)$ to $3f(x)f'(x)$. This is not a linear map; for example, $1 \mapsto 0$, $x \mapsto 3x$, but $x + 1 \mapsto 3(x + 1) = 3x + 3 \neq 3x + 0$. 

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