1111: Linear Algebra I

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Lecture 19

Linear maps

Definition 1. Suppose that V and W are two vector spaces. A function $\varphi\colon V \to W$ is said to be a *linear map*, or a *linear operator*, if

- for $v_1, v_2 \in V$, we have $\varphi(v_1 + v_2) = \varphi(v_1) + \varphi(v_2)$,
- for $c \in \mathbb{R}$, $v \in V$, we have $\varphi(c \cdot v) = c \cdot \varphi(v)$.

Lemma 1. *Suppose that φ is a linear map. Then $\varphi(0) = 0$, and $\varphi(-v) = -\varphi(v)$.*

Proof. This follows from $0 \cdot v = 0$ and $(-1) \cdot v = -v$. □

Definition 2. Let $\varphi\colon V \to W$ be a linear operator, and let $e_1, \ldots, e_n$ and $f_1, \ldots, f_m$ be bases of V and W respectively. Let us compute coordinates of the vectors $\varphi(e_i)$ with respect to the basis $f_1, \ldots, f_m$:

$$\begin{aligned}
\varphi(e_1) &= a_{11}f_1 + a_{21}f_2 + \cdots + a_{m1}f_m,\\
\varphi(e_2) &= a_{12}f_1 + a_{22}f_2 + \cdots + a_{m2}f_m,\\
&\cdots\\
\varphi(e_n) &= a_{1n}f_1 + a_{2n}f_2 + \cdots + a_{mn}f_m.
\end{aligned}$$

The matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \ldots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$$

is called *the matrix of the linear operator φ with respect to the given bases*, and denoted $A_{\varphi,\mathbf{e},\mathbf{f}}$. For each k, its k-th column is the column of coordinates of image $f(e_k)$.

Similarly to how we proved it for transition matrices, we have the following result.

Lemma 2. *Let $\varphi\colon V \to W$ be a linear operator, and let $e_1, \ldots, e_n$ and $f_1, \ldots, f_m$ be bases of V and W respectively. Suppose that $x_1, \ldots, x_n$ are coordinates of some vector $\mathbf{v}$ relative to the basis $e_1, \ldots, e_n$, and $y_1, \ldots, y_m$ are coordinates of $\varphi(\mathbf{v})$ relative to the basis $f_1, \ldots, f_m$. Then*

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = A_{\varphi,\mathbf{e},\mathbf{f}} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},$$

or, in other words,

$$(\varphi(\mathbf{v}))_{\mathbf{f}} = A_{\varphi,\mathbf{e},\mathbf{f}}\mathbf{v}_{\mathbf{e}}.$$

Proof. The proof is indeed very analogous to the one for transition matrices: we have

$$\mathbf{v} = x_1 e_1 + \cdots + x_n e_n,$$

so that

$$\varphi(\mathbf{v}) = x_1 \varphi(e_1) + \cdots + x_n \varphi(e_n).$$

Substituting the expansion of $f(e_i)$'s in terms of f_j's, we get

$$\varphi(\mathbf{v}) = x_1(a_{11}f_1 + a_{21}f_2 + \cdots + a_{m1}f_m) + \cdots + x_n(a_{1n}f_1 + a_{2n}f_2 + \cdots + a_{mn}f_m) = \\ = (a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n)f_1 + \cdots + (a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n)f_n.$$

Since we know that coordinates are uniquely defined, we conclude that

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = y_1, \\ \ldots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = y_n,$$

which is what we want to prove. □

The next statement is also similar to the corresponding one for transition matrices; it also generalises the statement that in the case of coordinate vector spaces product of matrices corresponds to composition of linear maps. In some sense, this is a central result about linear maps (which also justifies the definition of matrix products).

Theorem 1. *Let U, V, and W be vector spaces, and let $\psi\colon U \to V$ and $\varphi\colon V \to W$ be linear operators. Suppose that $e_1, \ldots, e_n$, $f_1, \ldots, f_m$, and $g_1, \ldots, g_k$ are bases of U, V, and W respectively. Let us consider the composite map $\varphi \circ \psi\colon U \to W$, $\varphi \circ \psi(u) = \varphi(\psi(u))$. Then*

1. *$\varphi \circ \psi$ is a linear map;*
2. *we have*

$$A_{\varphi\circ\psi,\mathbf{e},\mathbf{g}} = A_{\varphi,\mathbf{f},\mathbf{g}} A_{\psi,\mathbf{e},\mathbf{f}}.$$

Proof. First, let us note that

$$(\varphi \circ \psi)(u_1 + u_2) = \varphi(\psi(u_1 + u_2)) = \varphi(\psi(u_1) + \psi(u_2)) = \varphi(\psi(u_1)) + \varphi(\psi(u_2)) = (\varphi \circ \psi)(u_1) + (\varphi \circ \psi)(u_2), \\ (\varphi \circ \psi)(c \cdot u) = \varphi(\psi(c \cdot u)) = \varphi(c\psi(u)) = c\varphi(\psi(u)) = c(\varphi \circ \psi)(u),$$

so $\varphi \circ \psi$ is a linear map.

Let us prove the second statement. We take a vector $\mathbf{u} \in U$, and apply the formula of Lemma 2. On the one hand, we have

$$(\varphi \circ \psi(\mathbf{u}))_{\mathbf{g}} = A_{\varphi\circ\psi,\mathbf{e},\mathbf{g}} \mathbf{u}_{\mathbf{e}}.$$

On the other hand, we obtain,

$$(\varphi \circ \psi(\mathbf{u}))_{\mathbf{g}} = (\varphi(\psi(\mathbf{u})))_{\mathbf{g}} = A_{\varphi,\mathbf{f},\mathbf{g}}(\psi(\mathbf{u})_{\mathbf{f}}) = A_{\varphi,\mathbf{f},\mathbf{g}}(A_{\psi,\mathbf{e},\mathbf{f}} \mathbf{u}_{\mathbf{e}}) = (A_{\varphi,\mathbf{f},\mathbf{g}} A_{\psi,\mathbf{e},\mathbf{f}}) \mathbf{u}_{\mathbf{e}}.$$

Therefore

$$A_{\varphi\circ\psi,\mathbf{e},\mathbf{g}} \mathbf{u}_{\mathbf{e}} = (A_{\varphi,\mathbf{f},\mathbf{g}} A_{\psi,\mathbf{e},\mathbf{f}}) \mathbf{u}_{\mathbf{e}}$$

for every $\mathbf{u}_{\mathbf{e}}$. From our previous classes we know that knowing $A\mathbf{v}$ for all vectors $\mathbf{v}$ completely determines the matrix A, so

$$A_{\varphi\circ\psi,\mathbf{e},\mathbf{g}} = A_{\varphi,\mathbf{f},\mathbf{g}} A_{\psi,\mathbf{e},\mathbf{f}},$$

as required. □