Linear maps

**Definition 1.** Suppose that $V$ and $W$ are two vector spaces. A function $\varphi : V \rightarrow W$ is said to be a **linear map**, or a **linear operator**, if

- for $v_1, v_2 \in V$, we have $\varphi(v_1 + v_2) = \varphi(v_1) + \varphi(v_2)$,
- for $c \in \mathbb{R}$, $v \in V$, we have $\varphi(c \cdot v) = c \cdot \varphi(v)$.

**Lemma 1.** Suppose that $\varphi$ is a linear map. Then $\varphi(0) = 0$, and $\varphi(-v) = -\varphi(v)$.

**Proof.** This follows from $0 \cdot v = 0$ and $(-1) \cdot v = -v$. \(\square\)

**Definition 2.** Let $\varphi : V \rightarrow W$ be a linear operator, and let $e_1, \ldots, e_n$ and $f_1, \ldots, f_m$ be bases of $V$ and $W$ respectively. Let us compute coordinates of the vectors $\varphi(e_i)$ with respect to the basis $f_1, \ldots, f_m$:

$$
\varphi(e_1) = a_{11}f_1 + a_{21}f_2 + \cdots + a_{m1}f_m,
\varphi(e_2) = a_{12}f_1 + a_{22}f_2 + \cdots + a_{m2}f_m,
\vdots
\varphi(e_n) = a_{1n}f_1 + a_{2n}f_2 + \cdots + a_{mn}f_m.
$$

The matrix

$$
A = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
$$

is called the **matrix of the linear operator** $\varphi$ with respect to the **given bases**, and denoted $A_{\varphi,e,f}$. For each $k$, its $k$-th column is the column of coordinates of image $f(e_k)$.

Similarly to how we proved it for transition matrices, we have the following result.

**Lemma 2.** Let $\varphi : V \rightarrow W$ be a linear operator, and let $e_1, \ldots, e_n$ and $f_1, \ldots, f_m$ be bases of $V$ and $W$ respectively. Suppose that $x_1, \ldots, x_n$ are coordinates of some vector $v$ relative to the basis $e_1, \ldots, e_n$, and $y_1, \ldots, y_m$ are coordinates of $\varphi(v)$ relative to the basis $f_1, \ldots, f_m$. Then

$$
\begin{pmatrix}
y_1 \\
y_2 \\
\vdots \\
y_m
\end{pmatrix} = A_{\varphi,e,f} \begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix},
$$

or, in other words,

$$(\varphi(v))_f = A_{\varphi,e,f}v_e.$$
Proof. The proof is indeed very analogous to the one for transition matrices: we have

\[ v = x_1 e_1 + \cdots + x_n e_n, \]

so that

\[ \varphi(v) = x_1 \varphi(e_1) + \cdots + x_n \varphi(e_n). \]

Substituting the expansion of \( f(e_i) \)'s in terms of \( f_j \)'s, we get

\[
\varphi(v) = x_1(a_{11}f_1 + a_{21}f_2 + \cdots + a_{m1}f_m) + \cdots + x_n(a_{1n}f_1 + a_{2n}f_2 + \cdots + a_{mn}f_m) =
\]

\[
= (a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n)f_1 + \cdots + (a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n)f_n.
\]

Since we know that coordinates are uniquely defined, we conclude that

\[
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = y_1, \\
\cdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = y_n,
\]

which is what we want to prove. 

The next statement is also similar to the corresponding one for transition matrices; it also generalises the statement that in the case of coordinate vector spaces product of matrices corresponds to composition of linear maps. In some sense, this is a central result about linear maps (which also justifies the definition of matrix products).

**Theorem 1.** Let \( U, V, \) and \( W \) be vector spaces, and let \( \psi: U \to V \) and \( \varphi: V \to W \) be linear operators. Suppose that \( e_1, \ldots, e_n, f_1, \ldots, f_m, \) and \( g_1, \ldots, g_k \) are bases of \( U, V, \) and \( W \) respectively. Let us consider the composite map \( \varphi \circ \psi: U \to W, \) \( \varphi \circ \psi(u) = \varphi(\psi(u)). \) Then

1. \( \varphi \circ \psi \) is a linear map;
2. we have

\[ A_{\varphi \circ \psi, e, g} = A_{\varphi, f, g} A_{\psi, e, f}. \]

Proof. First, let us note that

\[
(\varphi \circ \psi)(u_1 + u_2) = \varphi(\psi(u_1 + u_2)) = \varphi(\psi(u_1) + \psi(u_2)) = \varphi(\psi(u_1)) + \varphi(\psi(u_2)) = (\varphi \circ \psi)(u_1) + (\varphi \circ \psi)(u_2), \\
(\varphi \circ \psi)(c \cdot u) = \varphi(\psi(c \cdot u)) = c \varphi(\psi(u)) = c(\varphi \circ \psi)(u),
\]

so \( \varphi \circ \psi \) is a linear map.

Let us prove the second statement. We take a vector \( u \in U, \) and apply the formula of Lemma 2. On the one hand, we have

\[
(\varphi \circ \psi(u))_g = A_{\varphi \circ \psi, e, g} u_e.
\]

On the other hand, we obtain

\[
(\varphi \circ \psi(u))_g = (\varphi(\psi(u)))_g = A_{\varphi, f, g} [\psi(u)_f] = A_{\varphi, f, g} (A_{\psi, e, f} u_e) = (A_{\varphi, f, g} A_{\psi, e, f}) u_e.
\]

Therefore

\[ A_{\varphi \circ \psi, e, g} u_e = (A_{\varphi, f, g} A_{\psi, e, f}) u_e \]

for every \( u_e. \) From our previous classes we know that knowing \( A v \) for all vectors \( v \) completely determines the matrix \( A, \) so

\[ A_{\varphi \circ \psi, e, g} = A_{\varphi, f, g} A_{\psi, e, f}, \]

as required. 

\[ \square \]