Dimension: examples

Example 1. The dimension of \( \mathbb{R}^n \) is equal to \( n \), as expected. (Standard unit vectors form a basis).

Example 2. The dimension of the space of polynomials in one variable \( x \) of degree at most \( n \) is equal to \( n + 1 \), since it has a basis \( 1, x, \ldots, x^n \).

Example 3. The dimension of the space of \( m \times n \)-matrices is equal to \( mn \). (Matrix units \( e_{ij} \), that is matrices that have the only nonzero element equal to 1, which is at the intersection of the \( i \)-th row and the \( j \)-th column, form a basis).

Example 4. For a matrix \( A \), the dimension of the solution space to the system of equations \( Ax = 0 \) is equal to the number of free unknowns, that is the number of columns of the reduced row echelon form of \( A \) that do not have pivots. (The spanning set we constructed previously forms a basis).

We also discussed in detail one of the tutorial questions — see the handout for the tutorial for that solution.

Change of coordinates

Let \( V \) be a vector space of dimension \( n \), and let \( e_1, \ldots, e_n \) and \( f_1, \ldots, f_n \) be two different bases of \( V \). Then we can compute coordinates of each vector \( v \) with respect to either of those bases, so that

\[
v = x_1 e_1 + \cdots + x_n e_n
\]

and

\[
v = y_1 f_1 + \cdots + y_n f_n.
\]

Our goal now is to figure out how these are related. For that, we shall need the notion of a transition matrix.

Definition 1. Let us express the vectors \( f_1, \ldots, f_n \) as linear combinations of \( e_1, \ldots, e_n \):

\[
f_1 = a_{11} e_1 + a_{21} e_2 + \cdots + a_{m1} e_m,
\]

\[
f_2 = a_{12} e_1 + a_{22} e_2 + \cdots + a_{m2} e_m,
\]

\[\vdots\]

\[
f_n = a_{1n} e_1 + a_{2n} e_2 + \cdots + a_{mn} e_m.
\]

The matrix \((a_{ij})\) is called the transition matrix from the basis \( e_1, \ldots, e_n \) to the basis \( f_1, \ldots, f_n \). Its \( k \)-th column is the column of coordinates of the vector \( f_k \) relative to the basis \( e_1, \ldots, e_n \).
Lemma 1. In the notation above, we have

\[
\begin{pmatrix}
    x_1 \\
    x_2 \\
    \vdots \\
    x_n \\
\end{pmatrix} =
\begin{pmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & \cdots & a_{nn} \\
\end{pmatrix}
\begin{pmatrix}
    y_1 \\
    y_2 \\
    \vdots \\
    y_n \\
\end{pmatrix}.
\]

In plain words, if we call \(e_1, \ldots, e_n\) the “old basis” and \(f_1, \ldots, f_n\) the “new basis”, then this system tells us that the product of the transition matrix with the columns of new coordinates of a vector is equal to the column of old coordinates.

Proof. The proof is fairly straightforward: we take the formula

\[ v = y_1 f_1 + \cdots + y_n f_n, \]

and substitute instead of \(f_i\)’s their expressions in terms of \(e_i\)’s:

\[
\begin{align*}
f_1 &= a_{11} e_1 + a_{21} e_2 + \cdots + a_{m1} e_m, \\
f_2 &= a_{12} e_1 + a_{22} e_2 + \cdots + a_{m2} e_m, \\
& \ \vdots \\
f_n &= a_{1n} e_1 + a_{2n} e_2 + \cdots + a_{mn} e_m.
\end{align*}
\]

What we get is

\[
y_1 (a_{11} e_1 + a_{21} e_2 + \cdots + a_{n1} e_n) + y_2 (a_{12} e_1 + a_{22} e_2 + \cdots + a_{n2} e_n) + \cdots + y_n (a_{1n} e_1 + a_{2n} e_2 + \cdots + a_{nn} e_n) = \\
(y_1 + y_2 + \cdots + y_n) (a_{11} e_1 + a_{21} e_2 + \cdots + a_{n1} e_n) = \\
\sum y_i (a_{ij} e_j) = \sum a_{ij} y_i e_j = \sum (a_{ij} y_i) e_j = x_1,
\]

which is what we want to prove.

If we denote, for a vector \(v\), the column of coordinates of \(v\) with respect to the basis \(e_1, \ldots, e_n\) by \(v_e\), and also denote the transition matrix from the basis \(e_1, \ldots, e_n\) to the basis \(f_1, \ldots, f_n\) by \(M_{e,f}\), then the previous result can be written as

\[ v_e = M_{e,f} v_f. \]

Lemma 2. We have

\[ M_{e,f} M_{f,g} = M_{e,g} \]

and

\[ M_{e,f} M_{f,e} = I_n \]

if \(\dim(V) = n\).

Proof. Applying the formula above twice, we have

\[ v_e = M_{e,f} v_f = M_{e,f} M_{f,g} v_g. \]

But we also have

\[ v_e = M_{e,g} v_g. \]

Therefore

\[ M_{e,f} M_{f,g} v_g = M_{e,g} v_g \]

for every \(v_g\). From our previous classes we know that knowing \(A v\) for all vectors \(v\) completely determines the matrix \(A\), so \(M_{e,f} M_{f,g} = M_{e,g}\) as required. Since manifestly we have \(M_{e,e} = I_n\), we conclude by letting \(g_k = e_k, k = 1, \ldots, n\), that \(M_{e,f} M_{f,e} = I_n\).