1111: Linear Algebra I

Dr. Vladimir Dotsenko (Vlad)

Lecture 14

Previously on...

A system of vectors is *linearly independent* in $\mathbb{R}^n$, if a nontrivial linear combination of these vectors cannot produce zero.

A system of vectors *spans* $\mathbb{R}^n$ (is *complete*) if every vector can be obtained as their linear combination.

A system of vectors is a *basis* of $\mathbb{R}^n$ if it is complete and linearly independent. This means that every vector can be obtained as their linear combination uniquely.

A *linear map* is a function from $\mathbb{R}^k$ to $\mathbb{R}^n$ which takes linear combinations into linear combinations (i.e. takes sums into sums and scalar multiples into scalar multiples). Every linear map can be obtained by multiplying vectors by a certain matrix.

Linear independence, span, and linear maps

Let $v_1, \ldots, v_k$ be vectors in $\mathbb{R}^n$. Consider the $n \times k$-matrix A whose columns are these vectors.

Let us relate linear independence and the spanning property to linear maps. We shall now show that

- the vectors $v_1, \ldots, v_k$ are linearly independent if and only if the map from $\mathbb{R}^k$ to $\mathbb{R}^n$ that send each vector x to the vector Ax is *injective*, that is maps different vectors to different vectors;
- the vectors $v_1, \ldots, v_k$ span $\mathbb{R}^n$ if and only if the map from $\mathbb{R}^k$ to $\mathbb{R}^n$ that send each vector x to the vector Ax is *surjective*, that is something is mapped to every vector b in $\mathbb{R}^n$.

Indeed, we can note that injectivity means that $Ax = b$ has at most one solution for each b, which is equivalent to the absence of free variables, which is equivalent to the system $Ax = 0$ having only the trivial solution, which we know to be equivalent to linear independence.
Also, surjectivity means that $Ax = b$ has solutions for every b, which we know to be equivalent to the spanning property.

Subspaces of $\mathbb{R}^n$

A non-empty subset U of $\mathbb{R}^n$ is called a *subspace* if the following properties are satisfied:

- whenever $v, w \in U$, we have $v + w \in U$;
- whenever $v \in U$, we have $c \cdot v \in U$ for every scalar c.

Of course, this implies that every linear combination of several vectors in U is again in U.

Let us give some examples. Of course, there are two very trivial examples: $U = \mathbb{R}^n$ and $U = \{0\}$.
The line $y = x$ in $\mathbb{R}^2$ is another example.
Any line or 2D plane containing the origin in $\mathbb{R}^3$ would also give an example, and these give a general intuition of what the word "subspace" should make one think of.
The set of all vectors with integer coordinates in $\mathbb{R}^2$ is an example of a subset which is NOT a subspace: the first property is satisfied, but the second one certainly fails.

Subspaces of $\mathbb{R}^n$: two main examples

Let A be an $m \times n$-matrix. Then the solution set to the homogeneous system of linear equations $Ax = 0$ is a subspace of $\mathbb{R}^n$. Indeed, it is non-empty because it contains $x = 0$. We also see that if $Av = 0$ and $Aw = 0$, then $A(v + w) = Av + Aw = 0$, and similarly if $Av = 0$, then $A(c \cdot v) = c \cdot Av = 0$.

Let $v_1, \ldots, v_k$ be some given vectors in $\mathbb{R}^n$. Their linear span $\mathrm{span}(v_1, \ldots, v_k)$ is the set of all possible linear combinations $c_1 v_1 + \ldots + c_k v_k$. The linear span of $k \geqslant 1$ vectors is a subspace of $\mathbb{R}^n$. Indeed, it is manifestly non-empty, and closed under sums and scalar multiples.

The example of the line $y = x$ from the previous slide fits into both contexts. First of all, it is the solution set to the system of equations $A\mathbf{x} = 0$, where $A = \begin{pmatrix} 1 & -1 \end{pmatrix}$, and $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$. Second, it is the linear span of the vector $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. We shall see that it is a general phenomenon: these two descriptions are equivalent.