1111: Linear Algebra I

Dr. Vladimir Dotsenko (Vlad)

Lecture 11

Previously on...

Row expansion for determinants:

$$\det(A) = a_{i1}C^{i1} + a_{i2}C^{i2} + \cdots + a_{in}C^{in},$$

In fact, we already encountered this in the case of 3×3-matrices. When studying vectors in 3D, we encountered the quantity $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ whose absolute value was shown to be equal to the volume of the parallelepiped built on the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. Note that we have

$$\mathbf{v} \times \mathbf{w} = (v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1),$$

where the coordinates are the first row cofactors of the matrix

$$A = \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} .$$

By inspection, we have $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \det(A)$.

Previously on...

Besides explaining conceptually why $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = -\mathbf{v}\cdot(\mathbf{u}\times\mathbf{w})$, the formula $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \det\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}$ actually brings a lot of useful insight. First, it allows to write a useful mnemonic formula for the cross product:

$$\mathbf{v}\times\mathbf{w} = \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix},$$

where one is supposed to expand the matrix along the first row:

$$\mathbf{v}\times\mathbf{w} = C^{11}\mathbf{i} + C^{12}\mathbf{j} + C^{13}\mathbf{k}.$$

It also suggests that the n-dimensional volume of the parallelepiped built on n vectors in the n-dimensional space must be equal in absolute value to the determinant whose rows (or columns) are these vectors. This becomes absolutely crucial for computing higher dimensional integrals.

Cramer's formula for systems of linear equations

We know that if A is invertible then $Ax = b$ has just one solution $x = A^{-1}b$. Let us plug in the formula for A^{-1} that we have:

$$x = A^{-1}b = \frac{1}{\det(A)}\,\mathrm{adj}(A)b \ .$$

When we compute $\mathrm{adj}(A)b = C^T b$, we get the vector whose k-th entry is

$$C^{1k}b_1 + C^{2k}b_2 + \ldots + C^{nk}b_n \ .$$

What does it look like? It looks like a k-th column expansion of some determinant, more precisely, of the determinant of the matrix A_k which is obtained from A by replacing its k-th column with b. (This way, the cofactors of that column do not change).

Theorem. (Cramer's formula) Suppose that $\det(A) \neq 0$. Then coordinates of the only solution to the system of equations $Ax = b$ are

$$x_k = \frac{\det(A_k)}{\det(A)} \ .$$

Summary of systems of linear equations

Theorem. Let A be an $n \times n$-matrix, and b a vector with n entries. The following statements are equivalent:

- (a) the *homogeneous* system $Ax = 0$ has only the trivial solution $x = 0$;
- (b) the reduced row echelon form of A is I_n;
- (c) $\det(A) \neq 0$;
- (d) the matrix A is invertible;
- (e) the system $Ax = b$ has exactly one solution.

Proof. In principle, to show that five statements are equivalent, we need to do a lot of work. We could, for each pair, prove that they are equivalent, altogether $5 \cdot 4 = 20$ proofs. We could prove that $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d) \Leftrightarrow (e)$, altogether 8 proofs. What we shall do instead is prove $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e) \Rightarrow (a)$, just 5 proofs.

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Proof. $(a) \Rightarrow (b)$: by contradiction, if the reduced row echelon form has a row of zeros, we get free variables.
$(b) \Rightarrow (c)$: follows from properties of determinants, elementary operations multiply the determinant by nonzero scalars.
$(c) \Rightarrow (d)$: proved in several different ways already.
$(d) \Rightarrow (e)$: discussed early on, if A is invertible, then $x = A^{-1}b$ is clearly the only solution to $Ax = b$.
$(e) \Rightarrow (a)$: by contradiction, if v a solution to $Ax = b$ and w is a nontrivial solution to $Ay = 0$, then $v + w$ is another solution to $Ax = b$.

Summary of systems of linear equations

A very important consequence (finite dimensional Fredholm alternative):

> *For an $n \times n$-matrix A, the system $Ax = b$ either has exactly one solution for every b, or has infinitely many solutions for some choices of b and no solutions for some other choices.*

In particular, to prove that $Ax = b$ *has* solutions for every b, it is enough to prove that $Ax = 0$ has only the trivial solution.

An example for the Fredholm alternative

Let us consider the following question:

> *Given some numbers in the first row, the last row, the first column, and the last column of an $n \times n$-matrix, is it possible to fill the numbers in all the remaining slots in a way that each of them is the average of its 4 neighbours?*

This is the "discrete Dirichlet problem", a finite grid approximation to many foundational questions of mathematical physics.