

Some standard types of linear algebra questions

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This list below presents standard skills you should have after taking a linear algebra course. It does not cover all possible types of questions you may encounter in the exam paper; however most of the questions either are of one of these types or can be reduced to one of these types. In addition, you should be able to formulate standard definitions from the course and solve simple theoretical problems (see homeworks for examples of such problems).

1. Determine the general solution to a system of linear equations. (Use the reduced row echelon form, or the Cramer's rule with ratios of determinants.)
2. Find out whether or not a matrix is invertible. (Use determinants, or reduced row echelon form.)
3. Find out whether the given permutation of $1, 2, \dots, n$ is even or odd. (Count inversions in this permutation.)
4. Compute the inverse matrix for a matrix. (Row operations on $(A | I)$, or cofactors and the adjoint matrix (don't forget to take the transpose!))
5. Determine a basis for the solution space of a linear system. (If we assign parameters u_1, \dots, u_k to the free variables, the general solution is a vector that can be represented in the form $u_1 \mathbf{e}_1 + \dots + u_k \mathbf{e}_k$ for some $\mathbf{e}_1, \dots, \mathbf{e}_k$; these vectors form a basis for the solution space.)
6. Determine a basis of the linear span of some collection of vectors. (Use reduced row/column echelon form; see the solutions to the sample paper.)
7. Compute the rank of a matrix. (Computing the number of leading 1's in the reduced row echelon form appears to be the easiest way.)
8. Find out whether or not a system of vectors is a basis of the given vector space. (A useful fact: for n vectors in an n -dimensional space, it is enough to prove one of two conditions (complete/linearly independent). Also, either of these two facts is equivalent to the fact that the determinant formed of the coordinates is not equal to zero.)

9. Determine the intersection of two subspaces. (Find a basis for each one and solve a linear system; see the solutions to the sample paper.)
10. Determine a basis/dimension for the sum of two subspaces. (The sum is spanned by the union of two bases; see some of the previous questions.)
11. Find out whether or not the given vector v belongs to the given subspace U . (The matrix whose row space forms a basis of U should have the same rank as the matrix obtained from it by adjoining a new row — the row of coordinates of v .)
12. Determine a basis of a vector space V relative to its subspace U . (Let A be a matrix whose rows form a basis of U , B — a matrix whose rows form a basis of V . Find the reduced row echelon form R for A . Write B under R and “reduce” it, making sure that the result B' does not have nonzero entries in the columns where R has leading 1's. Then find the reduced row echelon form R' for B' . Rows of R' form a basis of V relative to U .)
13. Determine the Jordan normal form of a matrix. (See handouts.)
14. Find an orthogonal basis for the given Euclidean space. (Gram–Schmidt orthogonalisation.)
15. Determine the signature of a symmetric bilinear form. (Method 1 (always works): Bring it to the signed sum of squares, like in Gelfand’s book, pp. 42–44 (don’t forget that the three numbers in the signature add up to the dimension of the vector space!). Method 2 (works when top left corner determinants are nonzero): count the number of positive and negative terms in the sequence $\frac{1}{\Delta_1}, \frac{\Delta_1}{\Delta_2}, \dots, \frac{\Delta_{n-1}}{\Delta_n}$ (in this case there are no zeros). Method 3 (not efficient for large matrices): count the number of positive, negative and zero eigenvalues of the matrix.)
16. Determine whether or not a quadratic form is positive definite. (Sylvester’s criterion.)

Let us consider an example that covers many of these types. Denote

$$A = \begin{pmatrix} -6 & -8 & -4 & -4 \\ 3 & 4 & 2 & 2 \\ 2 & 3 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

It is easy to see that $A^2 = 0$, so all the eigenvalues are all equal to zero. To determine the Jordan normal form, we should work with the subspace $\text{Ker}(A)$ of our vector space. Let us compute a basis of this subspace. To solve the system $Ax = 0$, we bring A to its reduced

row echelon form; it turns out to be $\begin{pmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Thus, the general solution to our

system is

$$\begin{pmatrix} -2u - 2v \\ u + v \\ u \\ v \end{pmatrix} = u \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + v \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

To find a Jordan basis of A , we should find a basis of the whole vector space relative to $\text{Ker}(A)$. We form a matrix $\begin{pmatrix} -2 & 1 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{pmatrix}$ whose rows contain coordinates of the basis vectors;

its reduced row echelon form is $\begin{pmatrix} 1 & -\frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -1 \end{pmatrix}$. Now, we take the matrix whose rows form a basis of the whole vector space; for such, we can take the identity matrix I . Reducing

the last four rows of the matrix $\begin{pmatrix} 1 & -\frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ by its first rows, we get $\begin{pmatrix} 1 & -\frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -1 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$;

now, computing the reduced row echelon form of the matrix formed by the last four rows, we

get $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, and its nonzero rows can be taken for a relative basis. Finally, we apply

A to each of the vectors v, w of the relative basis to get two threads of length 2, $v = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$,

$Av = \begin{pmatrix} -8 \\ 4 \\ 3 \\ 1 \end{pmatrix}$ and $w = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, $Aw = \begin{pmatrix} -4 \\ 2 \\ 1 \\ 1 \end{pmatrix}$; reversing the order of vectors in threads, we

obtain a Jordan basis Av, v, Aw, w ; relative to this basis, A has the matrix $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Exercise. How many types of questions from the list did we solve here?