Normal form of a general linear operator

Our goal now is, given a linear operator $A\colon V \to V$, to find a coordinate system on V relative to which the operator A has a matrix which is “simple enough” to understand the action of A on V in detail. Before we deal with this problem in full generality, let us consider two examples which show various subtleties of our problem.

Case $A^2 = A$

In this section we are dealing with a special case of linear operators, those satisfying $A^2 = A$.

Lemma 1. If $A^2 = A$, then $\mathrm{Im}(A) \cap \mathrm{Ker}(A) = \{0\}$.

Proof. Indeed, if $v \in \mathrm{Im}(A) \cap \mathrm{Ker}(A)$, then $v = Aw$ for some w, and $0 = Av = A(Aw) = A^2w = Aw = v$. □

Remark 1. From this proof, it is clear that if $v \in \mathrm{Im}(A)$, then $Av = v$.

Lemma 2. If $A^2 = A$, then $V = \mathrm{Im}(A) \oplus \mathrm{Ker}(A)$.

Proof. Indeed,

$$\dim(\mathrm{Im}(A)+\mathrm{Ker}(A)) = \dim \mathrm{Im}(A)+\dim \mathrm{Ker}(A)-\dim(\mathrm{Im}(A)\cap\mathrm{Ker}(A)) = \\ = \mathrm{rk}(A) + \dim \mathrm{Ker}(A) = \dim(V),$$

so the sum is a subspace of V of dimension equal to the dimension of V, that is V itself. Also, we already checked that the intersection is equal to 0, so the sum is direct. □

Consequently, if we take a basis of $\mathrm{Ker}(A)$, and a basis of $\mathrm{Im}(A)$, and join them together, we get a basis of V relative to which the matrix of A is $\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$, so the matrix of A is diagonal.

Case $A^2 = 0$

However nice the approach from the previous section seems, sometimes it does not work that well. Though we always have

$$\dim \mathrm{Im}(A) + \dim \mathrm{Ker}(A) = \dim(V),$$

the sum of these subspaces is not always direct, as the following example shows. If we know that $A^2 = 0$, that is $A(Av) = 0$ for every $v \in V$, that implies $\mathrm{Im}(A) \subset \mathrm{Ker}(A)$, so $\mathrm{Im}(A) + \mathrm{Ker}(A) = \mathrm{Ker}(A)$. We shall get back to this example later.

Finding a direct sum decomposition

Consider the sequence of subspaces $\mathrm{Ker}(A)$, $\mathrm{Ker}(A^2)$, ..., $\mathrm{Ker}(A^m)$, ...

Lemma 3. This sequence is increasing:

$$\mathrm{Ker}(A) \subset \mathrm{Ker}(A^2) \subset \ldots \subset \mathrm{Ker}(A^m) \subset \ldots$$

Proof. Indeed, if $v \in \mathrm{Ker}(A^s)$, that is $A^s v = 0$, we have

$$A^{s+1}v = A(A^s v) = 0.$$

□

Since we only work with finite-dimensional vector spaces, this sequence of subspaces cannot be strictly increasing; if $\mathrm{Ker}(A^i) \neq \mathrm{Ker}(A^{i+1})$, then, obviously, $\dim \mathrm{Ker}(A^{i+1}) \geqslant 1 + \dim \mathrm{Ker}(A^i)$. It follows that for some k we have $\mathrm{Ker}(A^k) = \mathrm{Ker}(A^{k+1})$.

Lemma 4. We have $\mathrm{Ker}(A^k) = \mathrm{Ker}(A^{k+1}) = \mathrm{Ker}(A^{k+2}) = \ldots$

Proof. Let us prove that $\mathrm{Ker}(A^{k+l}) = \mathrm{Ker}(A^{k+l-1})$ by induction on l. Note that the induction basis (case $l = 1$) follows immediately from our notation.Suppose that $\mathrm{Ker}(A^{k+l}) = \mathrm{Ker}(A^{k+l-1})$; let us prove that $\mathrm{Ker}(A^{k+l+1}) = \mathrm{Ker}(A^{k+l})$. Let us take a vector $v \in \mathrm{Ker}(A^{k+l+1})$, so $A^{k+l+1}v = 0$. We have $A^{k+l+1}v = A^{k+l}(Av)$, so $Av \in \mathrm{Ker}(A^{k+l})$. But by the induction hypothesis $\mathrm{Ker}(A^{k+l}) = \mathrm{Ker}(A^{k+l-1})$, so $A^{k+l-1}(Av) = 0$, or $A^{k+l}v = 0$, and we proved the induction step. □

Lemma 5. $\mathrm{Ker}(A^k) \cap \mathrm{Im}(A^k) = \{0\}$.

Proof. Indeed, assume there is a vector $v \in \mathrm{Ker}(A^k) \cap \mathrm{Im}(A^k)$. This means that $A^k(v) = 0$ and that there exists a vector w such that $v = A^k(w)$. It follows that $A^{2k}(w) = 0$, so $w \in \mathrm{Ker}(A^{2k})$. But from the previous lemma we know that $\mathrm{Ker}(A^{2k}) = \mathrm{Ker}(A^k)$, so $w \in \mathrm{Ker}(A^k)$. Thus, $v = A^k(w) = 0$, which is what we need. □

Lemma 6. $V = \mathrm{Ker}(A^k) \oplus \mathrm{Im}(A^k)$.

Proof. Indeed, consider the sum of these two subspaces (which is, as we just proved in the previous lemma, direct). It is a subspace of V of dimension $\dim \mathrm{Ker}(A^k) + \dim \mathrm{Im}(A^k) = \dim(V)$, so it has to coincide with V. □

Direct sums and block matrices

To proceed further, we need one new definition.

Definition 1. A subspace U of a vector space V is called an invariant subspace of a linear operator $A\colon V \to V$ if for any $u \in U$ we have $A(u) \in U$.

Two examples to keep in mind: 1) all multiples of some eigenvector form an invariant subspace, and 2) if $V = \mathbb{R}^3$, A is the rotation about the line l

through the angle α, then both l and its orthogonal 2d plane are invariant subspaces.

Let U_1 and U_2 be two subspaces of V such that $V = U_1 \oplus U_2$. If we fix a basis for U_1 and a basis for U_2 and join them together, we get a basis for V. When we write down the matrix of any linear operator with respect to this basis, we get a block-diagonal matrix $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$; its splitting into blocks correspond to the way our basis is split into two parts.

Let use formulate important facts which are immediate from the definition of a matrix of a linear operator.

1. $A_{12} = 0$ if and only if U_2 is invariant;
2. $A_{21} = 0$ if and only if U_1 is invariant.

Thus, this matrix is block-triangular if and only if one of the subspaces is invariant, and is block-diagonal if and only if both subspaces are invariant. This admits an obvious generalisation for the case of larger number of summands in the direct sum.

Reduction to the nilpotent case

Our next step in exploring general linear operators is decomposing V into a direct sum of invariant subspaces where our operator has only one eigenvalue. Let λ be an eigenvalue of A, and let us consider the operator $B = A - \lambda I$. Considering kernels of its powers, as in the previous sections, we find where they stabilise, and assume that $\mathrm{Ker}(B^k) = \mathrm{Ker}(B^{k+1}) = \ldots$.

Lemma 7. $\mathrm{Ker}(B^k)$ and $\mathrm{Im}(B^k)$ are invariant subspaces of A.

Proof. Indeed, note that $A(A - \lambda I) = (A - \lambda I)A$, so

– if $(A - \lambda I)^k(v) = 0$, then $(A - \lambda I)^k(A(v)) = A(A - \lambda I)^k(v) = 0$;
– if $v = (A - \lambda I)^k(w)$, then $A(v) = A(A - \lambda I)^k(w) = (A - \lambda I)^k(A(w))$.

□

To complete this step, we use induction on $\dim V$. Note that on the invariant subspace $\mathrm{Ker}(B^k)$ the operator A has only one eigenvalue, that is λ. (Indeed, if $Av = \mu v$ for some $0 \neq v \in \mathrm{Ker}(B^k)$, then $Bv = (A - \lambda I)v = (\mu - \lambda)v$, and $0 = B^k v = (\mu - \lambda)^k v$, so $\mu = \lambda$.) Also, on the invariant subspace $\mathrm{Im}(B^k)$ the operator A has no eigenvalues equal to λ. (Indeed, if $Av = \lambda v$ for some $0 \neq v \in \mathrm{Im}(B^k)$, then $Bv = (A - \lambda I)v = 0$, $B^k v = 0$, and we know that $\mathrm{Im}(B^k) \cap \mathrm{Ker}(B^k) = \{0\}$.)

The dimension of $\mathrm{Im}(A - \lambda\,\mathrm{Id})^k$ is less than $\dim V$ (if λ is an eigenvalue of A), so considering the linear operator A acting on the vector space

$V' = \mathrm{Im}(A - \lambda\,\mathrm{Id})^k$, we get the following result by induction on dimesnsion of V:

Theorem. For every linear operator $A\colon V \to V$ whose (different) eigenvalues are $\lambda_1, \ldots, \lambda_k$, there exist integers $n_1, \ldots, n_k$ such that

$$V = \mathrm{Ker}(A - \lambda_1\,\mathrm{Id})^{n_1} \oplus \ldots \oplus \mathrm{Ker}(A - \lambda_k\,\mathrm{Id})^{n_k}.$$

Normal form for a nilpotent operator

The second step in the proof is to establish the Jordan normal form theorem for the case of an operator $B\colon V \to V$ for which $B^k = 0$ (such operators are called nilpotent). This would basically complete the proof, after we put $B = A - \lambda\,\mathrm{Id}$ and use the result that we already obtained; we will discuss it more precisely later.

Let us modify slightly the notation we used in the previous section; put $N_1 = \mathrm{Ker}\,B$, $N_2 = \mathrm{Ker}\,B^2$, $\ldots$, $N_m = \mathrm{Ker}\,B^m$, $\ldots$ We denote by k the level at which stabilisation happens; we have $N_k = N_{k+1} = N_{k+2} = \ldots = V$.

Now we are going to prove our statement, constructing a required basis in k steps. First, find a basis of $V = N_k$ relative to N_{k-1}. Let $e_1, \ldots, e_s$ be vectors of this basis.

Lemma 8. The vectors $e_1, \ldots, e_s, B(e_1), \ldots, B(e_s)$ are linearly independent relative to N_{k-2}.

Indeed, assume that

$$c_1e_1 + \ldots + c_se_s + d_1B(e_1) + \ldots + d_sB(e_s) \in N_{k-2}.$$

Since $e_i \in N_k$, we have $B(e_i) \in N_{k-1} \supset N_{k-2}$, so

$$c_1e_1 + \ldots + c_se_s \in -d_1B(e_1) - \ldots - d_sB(e_s) + N_{k-2} \subset N_{k-1},$$

which means that $c_1 = \ldots = c_s = 0$ ($e_1, \ldots, e_s$ form a basis relative to N_{k-1}). Thus,

$$B(d_1e_1 + \ldots + d_se_s) = d_1B(e_1) + \ldots + d_sB(e_s) \in N_{k-2},$$

so

$$d_1e_1 + \ldots + d_se_s \in N_{k-1},$$

and we deduce that $d_1 = \ldots = d_s = 0$ ($e_1, \ldots, e_s$ form a basis relative to N_{k-1}), so the lemma follows.

Now we extend this collection of vectors by vectors $f_1, \ldots, f_t$ which together with $B(e_1), \ldots, B(e_s)$ form a basis of N_{k-1} relative to N_{k-2}. Absolutely analogously one can prove

Lemma 9. The vectors $e_1, \dots, e_s, B(e_1), \dots, B(e_s), B^2(e_1), \dots, B^2(e_s)$, $f_1, \dots, f_t, B(f_1), \dots, B(f_t)$ are linearly independent relative to N_{k-3}.

We continue that extension process until we end up with a usual basis of V of the following form:

$$\begin{gathered} e_1, \dots, e_s, B(e_1), \dots, B(e_s), B^2(e_1), \dots, B^{k-1}(e_1), \dots, B^{k-1}(e_s), \\ f_1, \dots, f_t, B(f_1), \dots, B^{k-2}(f_1), \dots, B^{k-2}(f_t), \\ \dots, \\ g_1, \dots, g_u, \end{gathered}$$

where the first line contains several "k-threads" formed by a vector from N_k, a vector from N_{k-1}, ..., a vector from N_1, the second line — several $(k-1)$-threads (a vector from N_{k-1}, a vector from N_{k-2}, ..., a vector from N_1), ..., the last line — several 1-threads, each being just a vector from N_1.

Remark 2. Note that if we denote by m_d the number of d-threads, we have

$$\begin{aligned} m_1 + m_2 + \ldots + m_k &= \dim N_1, \\ m_2 + \ldots + m_k &= \dim N_2 - \dim N_1, \\ &\cdots \\ m_k &= \dim N_k - \dim N_{k-1}, \end{aligned}$$

so the numbers of threads of various lengths are uniquely determined by the properties of our operator.

Example 1. If we assume, as in the beginning of this handout, that $A^2 = 0$, then all threads are of length 1 and 2, and we have a basis $e_1, f_1, e_2, f_2, \dots, e_m, f_m, g_1, g_2, \dots, g_k$ for which $Ae_1 = f_1$, $Af_1 = 0$, $Ae_2 = f_2$, $Af_2 = 0$, ..., $Ae_m = f_m$, $Af_m = 0$, $Ag_1 = 0$, ..., $Ag_k = 0$.

Normal form for a general linear operator

We know that V can be decomposed into a direct sum of invariant subspaces $\mathrm{Ker}(A-\lambda_i I)^{n_i}$; for each such subspace, the operator $B_i = A-\lambda_i I$ is nilpotent on it. Using the previous section, we deduce that each of these subspaces can be decomposed into a direct sum of subspaces where B acts by moving vectors along threads; numbers of threads of various lengths can be computed from the dimension data as above.

Summing up, we obtain the following theorem (which is usually called Jordan normal form theorem, or Jordan decomposition theorem):

Jordan normal form theorem. Let V be a finite-dimesional vector space. For a linear operator $A \colon V \to V$, there exist

- a decomposition of V

$$V = V_1 \oplus V_2 \oplus \ldots \oplus V_p$$

  into a direct sum of invariant subspaces of A;
- a basis $e_1^{(i)}, \ldots, e_{n_i}^{(i)}$ of V_i for each $i = 1, \ldots, p$ such that

$$(A - \lambda_i \operatorname{Id})e_1^{(i)} = 0; (A - \lambda_i \operatorname{Id})e_2^{(i)} \quad = e_1^{(i)}; \ldots (A - \lambda_i \operatorname{Id})e_{n_i}^{(i)} = e_{n_i-1}^{(i)}$$

  for some λ_i (which may coincide or be different for different i). Dimensions of these subspaces and numbers λ_i are determined uniquely up to re-ordering.

Note that the order of vectors in each thread reversed, in order to make the matrix of A consist of "Jordan blocks"

$$J_k(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & \lambda & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \lambda & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & 1 & 0 \\ 0 & 0 & 0 & 0 & \ldots & \lambda & 1 \\ 0 & 0 & 0 & 0 & \ldots & 0 & \lambda \end{pmatrix}$$

placed along the diagonal. Each block corresponds to a thread constructed above.