UNIVERSITY OF DUBLIN
TRINITY COLLEGE

Faculty of Science

school of mathematics

JF Mathematics **Trinity Term 2011**
JF Theoretical Physics
JF Two Subject Mod

Course 1212, a sample exam paper

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For each task, the number of points you can get for a complete solution of that task is printed next to it.

You may use all statements proved in class and in home assignments; when using some statement, you should formulate it clearly, e.g. "in class, we proved that if A is invertible, then the reduced row echelon form of A is the identity matrix".

All vector spaces unless otherwise specified are over complex numbers.

Non-programmable calculators are permitted for this examination.

1. In the vector space $V = \mathbb{R}^5$, consider the subspace U spanned by the vectors

$$\begin{pmatrix} 2 \\ 2 \\ 1 \\ 7 \\ -3 \end{pmatrix}, \quad \begin{pmatrix} -4 \\ 1 \\ -12 \\ 6 \\ -4 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 3 \\ 4 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 3 \\ 1 \\ 2 \end{pmatrix}, \text{ and } \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

   (a) (13 points) Compute $\dim U$.

   (b) (12 points) Which of the vectors $\begin{pmatrix} 4 \\ 0 \\ 5 \\ -3 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 1 \\ 8 \\ 4 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 4 \\ 2 \\ 4 \\ 0 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 0 \\ 5 \\ 0 \\ 2 \end{pmatrix}$ belong to U?

2. Consider the matrices

$$A = \begin{pmatrix} 2 & 3 & 4 \\ -2 & -2 & -2 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

   (a) (15 points) Describe the Jordan normal form and find some Jordan basis for A.

   (b) (15 points) Is A similar to B? Is A^2 similar to B? Explain your answers.

3. (a) (5 points) Write down the definition of a bilinear form on a real vector space. Which symmetric bilinear forms are said to be positive definite?

   (b) (15 points) Consider the vector space V of all polynomials in t of degree at most 2. The bilinear form ψ_a on V (depending on a [real] parameter a) is defined by the formula

$$\psi_a(f(t), g(t)) = \int_{-1}^{1} f(t)g(t)(t-a)\, dt.$$

   Determine all values of a for which ψ_a is positive definite.

4. Consider the vector space V of all $n \times n$-matrices, and define a bilinear form on this space by the formula $(A, B) = \mathrm{tr}(AB^T)$.

(a) (10 points) Show that this bilinear form actually defines a scalar product on the space of all matrices.

(b) (15 points) Show that with respect to that scalar product the subspace of all symmetric matrices (matrices A with $A = A^T$) is the orthogonal complement of the space of all skew-symmetric matrices (matrices A with $A = -A^T$).