

Solutions to the midterm test 2011

1. The rank of a linear operator $A: V \rightarrow W$ is defined to be equal to the dimension of image of A .

In class, we proved that rank of a matrix does not change when we perform elementary row and column operations. Bringing the matrix in question to its reduced row echelon form, we obtain the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -4/15 & -14/15 & 2/5 & -7/15 \\ 0 & 1 & 0 & 0 & -29/15 & -19/15 & -8/5 & 28/15 \\ 0 & 0 & 1 & 0 & 4 & 5 & 2 & -1 \\ 0 & 0 & 0 & 1 & -34/15 & -74/15 & -3/5 & -22/15 \end{pmatrix},$$

which manifestly has rank 4, since its first four columns are linearly independent.

2. Let us find a basis for the subspace U . For that, we shall perform elementary column operations on the matrix composed from the given vectors, bringing its to its reduced column echelon form. We obtain the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1/2 & -3 & 1/2 & 1/2 & 0 \end{pmatrix}.$$

Its nonzero columns form a basis of our vector space, which is therefore 4-dimensional. To check which of the given vectors belong to U , we should append each of those vectors to the matrix we are studying and see if we can reduce it completely using the basis vectors we obtained. Doing so, we find that the first, the second and the fourth vector belong to U , and the third vector does not belong to U .

3. The characteristic polynomial of A is $-t^3 + 3t^2 + 4 = -(t-1)(t+2)^2$, so the eigenvalues of A are 1 and -2 .

$$\text{Furthermore, } \text{rk}(A - I) = 2, \text{rk}((A - I)^2) = \text{rk} \begin{pmatrix} 15 & -6 & 0 \\ 6 & 3 & 0 \\ 6 & 3 & 0 \end{pmatrix} = 2,$$

$\text{rk}(A + 2I) = 2$, $\text{rk}(A + 2I)^2 = \text{rk} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -9 & 9 \end{pmatrix} = 1$. Thus, the kernels

of powers of $A - I$ stabilise instantly, so we should expect a thread of length 1 for the eigenvalue 1, whereas the kernels of powers of $A + 2I$ do not stabilise for at least two steps, so that would give a thread of length at least 2, hence a thread of length 2 because our space is 3-dimensional.

To determine the basis of $\text{Ker}(A - I)$, we solve the system $Av = 0$ and obtain a vector $f = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

To deal with the eigenvalue -2 , we see that the kernel of $A + 2I$ is spanned by the vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, the kernel of $(A + 2I)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -9 & 9 \end{pmatrix}$

is spanned by the vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. Reducing the latter vectors

using the former one, we end up with the relative basis vector $e = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$,

which gives rise to a thread $e, (A + 2I)e = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Overall, a Jordan

basis is given by $f, (A + 2I)e, e$, and the Jordan normal form has a block of size 2 with 1 on the diagonal, and a block of size 1 with 0 on the diagonal.

For the matrix B , all the above data is easy to compute because the matrix is already triangular. We observe that the eigenvalues are clearly the same as for A , and ranks of powers of $(B - I)$ and $(B + 2I)$ are the same as well, so these two matrices have the same Jordan form and therefore represent the same operator.