UNIVERSITY OF DUBLIN

XMA1132

TRINITY COLLEGE

FACULTY OF SCIENCE

SCHOOL OF MATHEMATICS

JF Mathematics
JF Theoretical Physics
JF Two Subject Mod

Hilary Term 2009

COURSE 113

Wednesday, March 11 Luce Hall 9.30–11.00

Dr. Vladimir Dotsenko

For each task, the number of points you can get for a complete solution of that task is printed next to it.

You may use all statements proved in class and in home assignments; when using some statement, you should formulate it clearly, e.g. "in class, we proved that if A is invertible, then the reduced row echelon form of A is the identity matrix".

Non-programmable calculators are permitted for this examination.

1. (a) (6 points) Under which condition a system of vectors of a vector space V is called complete? Prove that if a system of vectors is complete, then it remains complete after being extended by any vector $\mathbf{v}$ from V.

   (b) (8 points) Assume that the system of vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ (all belonging to the same vector space V) is complete. Prove that then the system of vectors $\mathbf{u}' = \mathbf{u} + \mathbf{v}$, $\mathbf{v}' = \mathbf{u} - \mathbf{w}$, $\mathbf{w}' = 2\mathbf{v} + \mathbf{w}$ is also complete. What are possible values of $\dim V$ in this situation? Explain your answer.

2. (a) (5 points) Define the rank of a linear operator.

   (b) (9 points) Show that for every vector $\mathbf{v} \in \mathbb{R}^3$ the mapping $A_{\mathbf{v}} \colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by the formula

   $$A_{\mathbf{v}}(\mathbf{w}) = \mathbf{v} \times \mathbf{w}$$

   is a linear operator, and show that for $\mathbf{v} \neq 0$ this operator has rank 2.

   (c) (12 points) Let U, V and W be three vector spaces. Show that for every two linear operators $A \colon V \to W$ and $B \colon U \to V$ we have

   $$\mathrm{rk}(AB) \leq \mathrm{rk}(A) \quad \text{and} \quad \mathrm{rk}(AB) \leq \mathrm{rk}(B).$$

3. Consider the matrices

   $$A = \begin{pmatrix} 9 & 5 & 2 \\ -16 & -9 & -4 \\ 2 & 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

   (a) (7 points) Describe all eigenvalues and eigenvectors of A and B.

   (b) (16 points) Describe the Jordan normal form of A and find a Jordan basis for A.

   (c) (8 points) Is A similar to B? Explain your answer.

   (d) (9 points) Find a closed formula for A^n.

4. (20 points) Assume that for a $n \times n$-matrix A with real matrix elements we have $A^2 = -E$. Prove that $\operatorname{tr} A = 0$.