For each task, the number of points you can get for a complete solution of that task is printed next to it.

You may use all statements proved in class and in home assignments; when using some statement, you should formulate it clearly, e.g. “in class, we proved that if $A$ is invertible, then the reduced row echelon form of $A$ is the identity matrix”.

Non-programmable calculators are permitted for this examination.
1. (a) (6 points) Under which condition a system of vectors of a vector space $V$ is called complete? Prove that if a system of vectors is complete, then it remains complete after being extended by any vector $v$ from $V$.

(b) (8 points) Assume that the system of vectors $u$, $v$, and $w$ (all belonging to the same vector space $V$) is complete. Prove that then the system of vectors $u' = u + v$, $v' = u - w$, $w' = 2v + w$ is also complete. What are possible values of $\dim V$ in this situation? Explain your answer.

2. (a) (5 points) Define the rank of a linear operator.

(b) (9 points) Show that for every vector $v \in \mathbb{R}^3$ the mapping $A_v : \mathbb{R}^3 \to \mathbb{R}^3$ defined by the formula

$$A_v(w) = v \times w$$

is a linear operator, and show that for $v \neq 0$ this operator has rank 2.

(c) (12 points) Let $U$, $V$ and $W$ be three vector spaces. Show that for every two linear operators $A : V \to W$ and $B : U \to V$ we have

$$\text{rk}(AB) \leq \text{rk}(A) \quad \text{and} \quad \text{rk}(AB) \leq \text{rk}(B).$$

3. Consider the matrices

$$A = \begin{pmatrix} 9 & 5 & 2 \\ -16 & -9 & -4 \\ 2 & 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) (7 points) Describe all eigenvalues and eigenvectors of $A$ and $B$.

(b) (16 points) Describe the Jordan normal form of $A$ and find a Jordan basis for $A$.

(c) (8 points) Is $A$ similar to $B$? Explain your answer.

(d) (9 points) Find a closed formula for $A^n$.

4. (20 points) Assume that for a $n \times n$-matrix $A$ with real matrix elements we have $A^2 = -E$. Prove that $\text{tr} \ A = 0$. 

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